Learning to predict complex outputs

A kernel view

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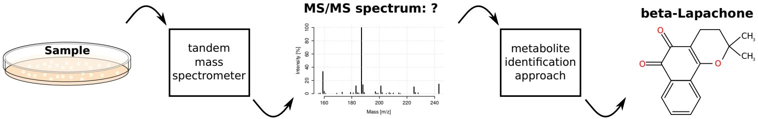
Motivation: Metabolite identification using mass spectrometry

Goal: help chemists to identify metabolites in a biological sample using mass spectra.



Sample

tandem mass spectrometer

MS/MS spectrum: ?

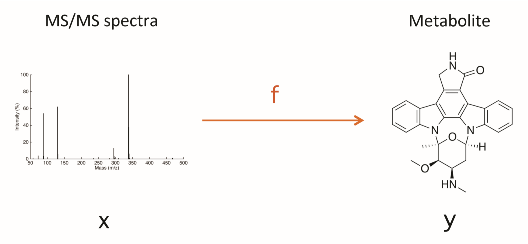
metabolite identification approach

beta-Lapachone

(Dührkop et al., 2015, Nguyen et al. 2018)

Supervised metabolite prediction from mass spectra

Assume we observe pairs of mass spectra and graphs, use them to train a labeled graph prediction model



(Brouard et al. 2016, Brouard et al. 2019)

Proposition of a generic framework

- Choose an appropriate representation vector space $\mathcal{Z}$ for complex outputs
- Regress the output in this representation space $\mathcal{Z}$ especially by leveraging regularization and get $\hat{h} : \mathcal{X} \to \mathcal{Z}$
- Structured prediction: at testing time, solve a pre-image problem and get $\hat{f} : \mathcal{Z} \to \mathcal{Y}$ by decoding $\hat{f} = d \circ \hat{h}$

In this talk, focus on:

- Learning functions with values in a Hilbert space $\mathcal{Z}$
- $\mathcal{Z}$ is chosen to be a Reproducing Kernel Hilbert Space associated to a so-called output kernel, i.e. a similarity between outputs.

Strong links with Functional Output Regression and Infinite Task learning

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Structured prediction

How to learn a function $f : \mathcal{X} \to \mathcal{Y}$ able to predict Y given X using n independently identically distributed data $(x_i, y_i)_{i=1}^n$ when $\mathcal{Y}$ is finite and huge ? (Nowozin and Lampert 2011)

Structured prediction

Learning problem

Given some loss function $\Delta : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$, the true risk structured prediction problem writes as:

$$\min_{f \in \mathcal{F}(\mathcal{X},\mathcal{Y})} \mathbb{E}_{X,Y}[\Delta(Y, f(X))]. \tag{1}$$

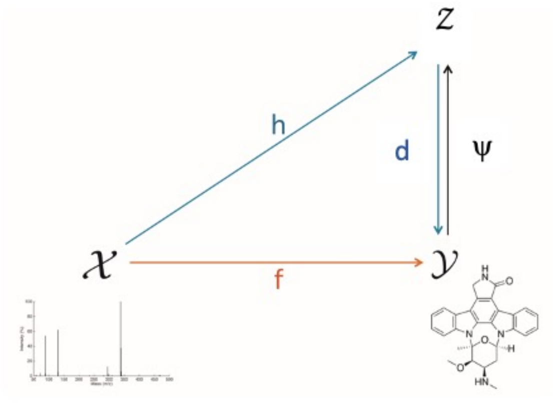
In supervised learning, we aim at finding a good estimator f_n of a minimizer of this problem using a given sample i.i.d. $\{(x_i, y_i)_{i=1}^n\})$.
N.B.: Δ should take into account the structured nature of objects in $\mathcal{Y}$.

Structured prediction

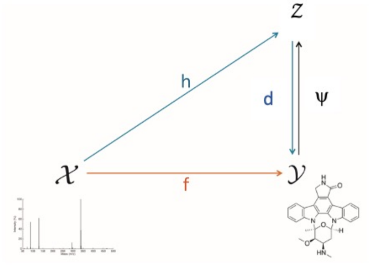
Various relaxations to mitigate the discrete nature of the output space (remind that $\mathcal{Y}$ is in general very large)

1. energy-based methods that learn an **energy function** g of input-output pairs so that: $f(x) = \arg\max_{y \in \mathcal{Y}} g(x, y)$ (Le Cun, 2006, ..., Bellanger et al. 2016)
2. end-to-end learning (Chen et al. 2015, Pierce et al. 2021, Pilula et al. 2018, Blondel ..) that learn to generate y or by softenizing ”arg max”
3. surrogate regression methods that embed the output into a Hilbert space and solve a **surrogate regression** problem: **output kernel regression** (KDE - Cortes et al. 2005, OK3 - Geurts et al. 2006, 07; IOKR- Brouard et al. 2011, Kadri et al. 13, Ciliberto et al. 2016)

Get the intuition with molecule identification from mass spectra



Novel problems to solve



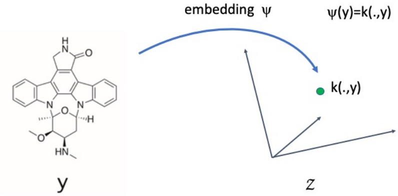
1. Define $\mathcal{Z}$ and $\psi : \mathcal{Y} \to \mathcal{Z}$
2. Learn $h : \mathcal{X} \to \mathcal{H}_{k_\mathcal{Y}}$ to predict $\psi(y)$ given x
3. Solve a pre-image problem : compute $f(x) = d \circ h(x)$.

Output Kernel Regression: use kernel trick in the output space

Choose a kernel $k_y : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ that encodes the similarity between structured objects

Take $\psi(y) = k(\cdot, y)$

$\mathcal{Z} := \mathcal{H}_{k_\mathcal{Y}}$



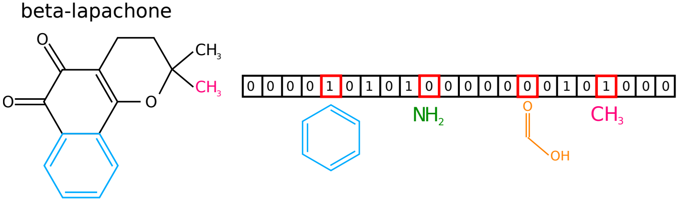
Advantages of defining a kernel $k(y, y')$ i?

- Allowing **infinite dimensional embeddings** while leveraging the kernel trick
- One principle to rule them all: kernels for various structured objects (See Gaertner 2006), opening the door to many structured tasks
  - label ranking (see Korba et al. 2018)
  - link prediction (Geurts et al. 2006, 2007)
  - image completion (Cortes et al. 2005, ...)
  - graph prediction (Brouard et al. 2020, Brogat-Motte et al. 2021)

A constraint however: to benefit from the kernel trick, not all the losses are suitable !

Example: kernel between molecules

Based on FingerID [Heinonen et al., 2012; Dührkop et al., 2015; Nguyen et al., 2018]



- Use **molecular fingerprint** $c(y) \in \mathbb{R}^d$ to encode the structure of a molecule as a (very large) binary vector
- Each entry indicates the existence or the frequency of a certain molecular property: atom or bond type, substructure (e.g. aromatic ring).

Use a Gaussian kernel on $c(y)$: $k_{\mathcal{Y}}(y, y') = \exp(-\gamma \|c(y) - c(y')\|^2)$

Structured prediction with Output Kernel Regression

Take $\Delta(y, y') = \ell(\psi(y), \psi(y'))$ and replace the target problem in Eq.1 by the surrogate problem:

$$\min_{h:\mathcal{X}\to\mathcal{Z}} \mathbb{E}_{X,Y}[\ell(\psi(Y), h(X))].$$

Empirical (regularized) counterpart: with $\Omega : \mathcal{H} \to \mathbb{R}^+$ and $\lambda > 0$ given some hypothesis space $\mathcal{H}$,

$$\min_{h\in\mathcal{H}} \frac{1}{n}\sum_{i=1}^{n} \ell(\psi(y_i), h(x_i)) + \lambda\Omega(h),$$

using a given dataset $\{(x_i, y_i)_{i=1}^{n}\}$.

Once we get h_n, define

$f_n(x) = d \circ h_n(x) = \arg\min_{y\in\mathcal{Y}} \ell(\psi(y), h_n(x))$

From a practical point of view

One wishes to use the kernel trick...

- Condition 1: ℓ must be computed by using inner products $\langle \psi(y), \psi(y') \rangle_{\mathcal{H}_{k_{\mathcal{Y}}}} = k(y, y')$.
- Condition 2: if an estimated model h_n writes as:

$$h_n(x) = \sum_{i=1}^{n} \beta_i(x) \psi(y_i)$$

with $\beta : \mathcal{X} \to \mathbb{R}^d$, then if ℓ satisfies Condition 1, one can compute $f_n(x) = \arg\min_{y \in \mathcal{Y}} \ell(\psi(y), h_n(x))$.

Non-parametric models come to the place: trees-based approaches, k-nearest-neighbors, ... , kernel methods

Example: Kernelized Regression Trees (OK3)

$$h_n^{tree}(x) = \sum_{i=1}^{n} \underbrace{\frac{1_i(t(x))}{\sum_{j=1}^{n} 1_j(t(x))}}_{\beta_i(x)} \psi(y_i),$$

with $t(x)$ is the index of the leaf where x falls, $1_i(p) = 1$ if x_i falls in leaf p, 0 otherwise.

- **Loss**: squared loss
- A split is thus selected by maximizing the reduction of embedded outputs' empirical variance
- Computations use kernel trick in the output space

Combined with ensemble methods, SOTA results [Geurts et al. 2006, 2007].

Now, from a theoretical point of view

Minimize the surrogate true risk:

$$h^* \in \arg\min_{h \in \mathcal{H}} \mathbb{E}_{X,Y}[\ell(\psi(Y), h(X))]$$

Decoding (pre-image):

$$d \circ h^*(x) = \arg\min_{y \in \mathcal{Y}} \ell(\psi(y), h^*(x))$$

Let f^* be a minimizer of the true target risk. We want to compare $d \circ h^*$ with f^*.

Output Kernel Regression fits the SELF and ILE framework

Structured Encoding Loss Function (SELF, Ciliberto et al. 2016), Nowak-Villa (2018, 2019), Luige et al. 2019, and Consistent Structured prediction with Implicit Loss Embeddings (2020):

- general conditions on $\mathcal{Y}$ and losses to get Fisher consistency and excess risk bounds

SELF property and consequences

Definition (SELF loss - Ciliberto et al. 2016)
$\Delta : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ is said to be SELF if it exists a separable Hilbert space $\mathcal{F}$, a feature map $\phi : \mathcal{Y} \to \mathcal{F}$ and a bounded linear operator A on $\mathcal{F}$ such that:

$$\Delta(y, y') = \langle \phi(y), A\phi(y') \rangle_{\mathcal{F}}$$

Theorem (Ciliberto et al. 2016)
Let Δ satisfy the SELF property with $\mathcal{Y}$ compact then, for every measurable function $h : \mathcal{X} \to \mathcal{F}$ and $d : \mathcal{F} \to \mathcal{Y}$, satisfying $d(z) = \arg\min_{y \in \mathcal{Y}} \langle \phi(y), Az \rangle_{\mathcal{F}}$, we have:

$$\epsilon(d \circ h^*) = \epsilon(f^*)$$
$$\epsilon(d \circ h) - \epsilon(f^*) \leq 2c_\Delta \sqrt{R(h) - R(h^*)},$$

with $\epsilon(f) = \mathbb{E}[\Delta(Y, f(X))] = \mathbb{E}[\langle \phi(y), A\phi(y') \rangle_{\mathcal{F}}]$ and $R(h) = \mathbb{E}[\|h(X) - \phi(Y)\|^2_{\mathcal{F}}]$

Output Kernel Regression - squared loss - fits the SELF framework

Trivial case: $k(y, y) = 1$ and $\ell(\psi(y), h(x)) = \|\psi(y) - h(x)\|^2_{\mathcal{H}_{k_\mathcal{Y}}}$.
Then :

$$\begin{aligned} f(x) &= d \circ h(x) \\ &= \arg\min_y \|\psi(y) - h(x)\|^2_{\mathcal{H}_{k_\mathcal{Y}}} \\ &= \arg\min_y - < \psi(y), h(x) > \end{aligned}$$

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Kernel-based approach

Use kernel in the input space as well, aka $\mathcal{H}$ is also a Reproducing Kernel Hilbert Space

Main Advantages:

kernels leverage convex optimization, govern regularization, allow for structured data in the input space as well.

Side advantages:

extend easily to semi-supervised learning

OK, but to get functions with values in Hilbert space $\mathcal{Z}$:

we need Operator-Valued Kernels (OVK) !

Operator-valued Kernels and vector-valued Reproducing Kernel Hilbert Spaces

- (Pedrycs, 1957) theory of vv-RKHS
- (Senkene and Tempel'man, 1973) theory of vv-RKHS
- (Hein and Bousquet, 2004) survey on positive definite kernels, including a short introduction to OVK
- (Micchelli and Pontil, 2005) learning vector-valued functions with OVK
- (Carmeli et al., 2006) theory of vv-RKHS
- (Carmeli et al. 2010) vv-RKHS and universality

From scalar-valued kernels to operator-valued kernels

Notations: if $\mathcal{Z}$ is a Hilbert Space, $\mathcal{L}(\mathcal{Z})$ is the space of bounded linear operators on $\mathcal{Z}$.

Scalar kernel	Operator-valued kernel
$k(x, x') \in \mathbb{R}$	$\mathcal{K}(x, x') \in \mathcal{L}(\mathcal{Z})$
$k(x, x') = k(x', x)$	$\mathcal{K}(x, x') = \mathcal{K}(x', x)^*$
$\forall (x_i, z_i)_{i=1}^m \in (\mathcal{X} \times \mathbb{R})^m,$	$\forall (x_i, z_i)_{i=1}^m \in (\mathcal{X} \times \mathcal{Z})^m,$
$\sum_{i,j=1}^m z_i z_j k(x_i, x_j) \geq 0$	$\sum_{i,j=1}^m \langle z_i, \mathcal{K}(x_i, x_j) z_j \rangle_{\mathcal{Z}} \geq 0$
$\mathcal{H}_k = \overline{Span\,\{k(\cdot, x), x \in \mathcal{X}\}}$	$\mathcal{H}_{\mathcal{K}} = \overline{Span\,\{\mathcal{K}(\cdot, x) z : x, z \in \mathcal{X} \times \mathcal{Z}\}}$
$\langle f, k(\cdot, x) \rangle_{\mathcal{H}_k} = f(x)$	$\langle f, \mathcal{K}(\cdot, x) z \rangle_{\mathcal{H}_{\mathcal{K}}} = \langle f(x), z \rangle_{\mathcal{Z}}$

Hint: think about the matrix-valued case

$\mathcal{Z} = \mathbb{R}^d$

- A trivial kernel : $\mathcal{K}(x, x') = I_{\mathcal{Z}}.k(x, x')$, where $I_{\mathcal{Z}}$ is the $d \times d$ identity matrix (independent outputs)
- A separable kernel: $\mathcal{K}(x, x') = A.k(x, x')$ where A is positive semi-definite matrix $d \times d$ (dependencies between outputs)

Important! As in scalar kernel methods, choosing $\mathcal{K}$ implies choosing the way you want to regularize when using $\| \cdot \|_{\mathcal{H}_{\mathcal{K}}}$

Separable Operator-valued kernels

In particular, we will make use of a special separable operator-valued kernel:

$$K(x, x') = I_{\mathcal{H}_{\mathcal{Y}}} k(x, x'),$$

N.B. If k is a universal kernel then K is universal.

More about operator-valued-kernels

Again general case: $\mathcal{Z}$ Hilbert Space

Scalar kernel	Operator-valued kernel
Representer theorem:	Representer Theorem:
$\min_{h\in\mathcal{H}_k} L(h(x_1),\dots,h(x_n)) + \lambda\|h\|^2_{\mathcal{H}_k}$	$\min_{h\in\mathcal{H}_\mathcal{K}} L(h(x_1),\dots,h(x_n)) + \lambda\|h\|^2_{\mathcal{H}_\mathcal{K}}$
$h_n(x) = \sum_{i=1}^n k(x,x_i)\alpha_i \in \mathbb{R}$	$h_n(x) = \sum_{i=1}^n \mathcal{K}(x,x_i)\alpha_i \in \mathcal{Z}$

N.B. A representer theorem for OVK but still we do not know how to compute $\alpha_i \in \mathcal{Z}$

A simple case: kernel ridge regression 1/2

Assume we observe $(x_i, z_i)_{i=1}^n$, define an operator-valued kernel
$\mathcal{K} : \mathcal{X} \times \mathcal{X} \to \mathcal{L}(\mathcal{Z})$ such that: $K(x, x') = Id_{\mathcal{Z}} k_X(x, x')$
Let us consider, for $\lambda > 0$:

$$\min_{h \in \mathcal{H}_{\mathcal{K}}} \sum_{i=1}^{n} \|z_i - h(x_i)\|_{\mathcal{Z}}^2 + \lambda \|h\|_{\mathcal{H}_{\mathcal{K}}}^2 \tag{2}$$

A simple case: kernel ridge regression 2/2

- The representer theorem (Micchelli and Pontil, 2005) applies
- The unique minimizer h_n writes: $h_{ridge}(x) = \sum_{i=1}^{n} \mathcal{K}(x, x_i)\hat{\alpha}_i$

where $\hat{\alpha}_i$'s enjoy a closed form, yielding to the following expression:

$$h_{ridge}(x) = \sum_{j=1}^{n} \beta_j(x) z_j, \tag{3}$$

with: $\beta(x) = (K_x + n\lambda I)^{-1} \kappa_X^x$
and $\kappa_X^x = [k_X(x_1, x), \ldots, k_X(x_n, x)]^T$.

Back to structured prediction: Input Output Kernel Ridge Regression (ridge-IOKR)

Now the feature space $\mathcal{Z} := \mathcal{H}_{k_\mathcal{Y}}$ is the RKHS associated to $k_\mathcal{Y}$, a kernel on $\mathcal{Y}$.
Define the OVK $K(x, x') = Id_{\mathcal{H}_{k_\mathcal{Y}}} k_X(x, x')$
Denote $\psi(y) = k_\mathcal{Y}(\cdot, y)$.

$$h_n(x) = \sum_{i=1}^{n} \beta_i(x)\psi(y_i), \tag{4}$$

with: $\beta(x) = (K_x + n\lambda I)^{-1}\kappa_X^x$
and $\kappa_X^x = [k_X(x_1, x), \dots, k_X(x_n, x)]^T$ and $\lambda > 0$.
Then, we are able to compute

$$f_n(x) = \arg\min_{y \in \mathcal{Y}} \|\psi(y_i) - h(x_i)\|^2_{\mathcal{H}_{k_\mathcal{Y}}}, \tag{5}$$

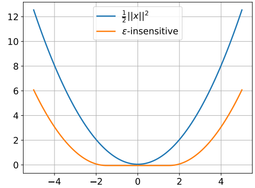
using only inner products of $\psi(y_i)$s.
NB. We retrieve Kernel Dependency Estimation of Cortes et al. as well.

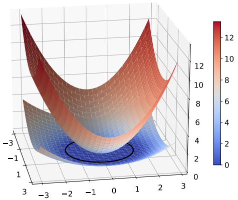
More about kernel ridge regression with input and output kernels

- Leveraging unlabeled input data: semi-supervised IOKR (ridge or not) - Brouard et al. 2011,16 with nice applications to link prediction.
- Leveraging structure in the output feature space: reduced-rank approach *Work of Luc Brogat-Motte et al., submitted*

Now more interesting loss functions: sparsity and robustness

ϵ-Ridge

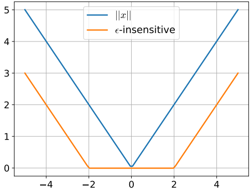


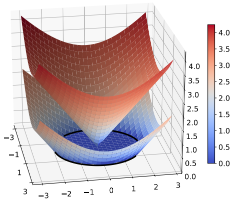


$$\frac{1}{2}\|\cdot\|^2 \,\square\, \chi_{\mathcal{B}_\epsilon}$$

(Sparsity)

ϵ-SVR

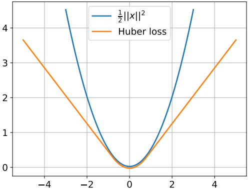


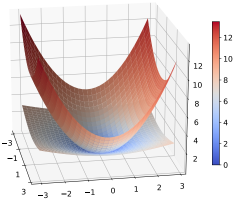


$$\|\cdot\| \,\square\, \chi_{\mathcal{B}_\epsilon}$$

(Sparsity, Robustness)

κ-Huber





$$\kappa\|\cdot\| \,\square\, \frac{1}{2}\|\cdot\|^2$$

(Robustness)

Data-sparse and robust loss: the example of ϵ-insensitive loss

Data-sparse and Robust losses [Sangnier et al. 2017, Laforgue et al. 2020]:

With a slight abuse of notation

Let $\ell : \mathcal{Z} \to \mathbb{R}$ be a convex loss with unique minimum

at 0, and $\epsilon > 0$. The ϵ-insensitive version of ℓ, denoted ℓ_ϵ, is defined by:

$$\ell_\epsilon(z) = \left(\ell \,\square\, \chi_{\mathcal{B}_\epsilon}\right)(z) = \begin{cases} \ell(0) & \text{if } \|z\|_{\mathcal{Z}} \leq \epsilon \\ \inf\limits_{\|d\|_{\mathcal{Z}} \leq 1} \ell(z - \epsilon d) & \text{otherwise} \end{cases},$$

Infimal convolution: $(f \square g)(x) = \inf_{x'} f(x') + g(x - x')$. (Bauschke et al. 2011)

Reminder: representer theorem and convex losses

General case: the output space is $\mathcal{Z}$: Hilbert Space and output training data are denoted z_i. Let $\ell : \mathcal{Z} \to \mathbb{R}$ a convex loss.

Theorem (Micchelli et Pontil 2005)

The solution to the learning problem is given by

$$h_n = \frac{1}{\lambda n} \sum_{i=1}^{n} \mathcal{K}(\cdot, x_i)\hat{\alpha}_i, \tag{6}$$

with $(\hat{\alpha}_i)_{i=1}^n \in \mathcal{Z}^n$ the solutions to the dual problem:

Problem

(Brouard et al. 2016, Sangnier et al. 2017)

$\min_{(\alpha_i)_{i=1}^n \in \mathcal{Z}^n} \ \sum_{i=1}^n \ell_i^\star(-\alpha_i) + \frac{1}{2\lambda n} \sum_{i,j=1}^n \langle \alpha_i, \mathcal{K}(x_i, x_j)\alpha_j \rangle_{\mathcal{Z}}$,

where $g^\star : \alpha \in \mathcal{Z} \mapsto \sup_{z \in \mathcal{Z}} \langle \alpha, z \rangle_{\mathcal{Z}} - g(z)$ denotes the Fenchel-Legendre transform of a function $g : \mathcal{Z} \to \mathbb{R}$.

with $\ell_i(y) = \ell(y_i - y)$.

Some limitations

- **1st limitation:** the Fenchel-Legendre transform $\ell^\star$ needs to be computable ($\rightarrow$ assumption)
- **2nd limitation :** the dual variables $(\alpha_i)_{i=1}^n$ are still **infinite dimensional!**

Some limitations

- **1st limitation:** the Fenchel-Legendre transform $\ell^\star$ needs to be computable ($\rightarrow$ assumption)
- **2nd limitation :** the dual variables $(\alpha_i)_{i=1}^n$ are still **infinite dimensional!**

If $\mathbf{Z} = \mathrm{Span}\{z_j,\ j \leq n\}$ invariant by $\mathcal{K}$, *i.e.*
$\forall (x, x'), \quad z \in \mathbf{Z} \ \Rightarrow \ \mathcal{K}(x, x')z \in \mathbf{Z}$

$$\hat{\alpha}_i \in \mathbf{Z} \quad \rightarrow \quad \text{possible reparametrization}$$

The double representer theorem

Laforgue et al. ICML 2020.

Theorem (Double representer theorem)

Assume that OVK $\mathcal{K}$ and loss ℓ satisfy the appropriate assumptions (see paper for details, verified by standard kernels and our losses), then

$\hat{h} = \underset{\mathcal{H}_{\mathcal{K}}}{\operatorname{argmin}} \ \frac{1}{n}\sum_i \ell(h(x_i) - z_i) + \frac{\lambda}{2}\|h\|^2_{\mathcal{H}_{\mathcal{K}}}$ *is given by*

$$\hat{h} = \frac{1}{\lambda n}\sum_{i,j=1}^{n} \mathcal{K}(\cdot, x_i)\ \hat{\omega}_{ij}\ z_j,$$

with $\hat{\Omega} = [\hat{\omega}_{ij}] \in \mathbb{R}^{n\times n}$ *the solution to the* ***finite dimensional*** *problem*

$$\min_{\Omega\in\mathbb{R}^{n\times n}} \ \sum_{i=1}^{n} L_i\left(\Omega_{i:}, K^Z\right) + \frac{1}{2\lambda n}\mathbf{Tr}\left(\tilde{M}^\top(\Omega\otimes\Omega)\right),$$

with $\tilde{M}$ *the* $n^2 \times n^2$ *matrix writing of* M *s.t.* $M_{ijkl} = \langle z_k, \mathcal{K}(x_i, x_j) z_l\rangle_{\mathcal{Z}}$.

Specific dual problems for our losses 1

If $\mathcal{K} = k\ \mathbf{I}_{\mathcal{Z}}$, the solutions to the ϵ-Ridge regression, κ-Huber regression, and ϵ-SVR primal problems

$$
\begin{aligned}
&(P1) \quad \min_{h \in \mathcal{H}_{\mathcal{K}}} \quad \frac{1}{2n} \sum_{i=1}^{n} \| h(x_i) - z_i \|_{\mathcal{Z},\epsilon}^2 + \frac{\Lambda}{2} \| h \|_{\mathcal{H}_{\mathcal{K}}}^2 , \\
&(P2) \quad \min_{h \in \mathcal{H}_{\mathcal{K}}} \quad \frac{1}{n} \sum_{i=1}^{n} \ell_{H,\kappa}(h(x_i) - z_i) + \frac{\Lambda}{2} \| h \|_{\mathcal{H}_{\mathcal{K}}}^2 , \\
&(P3) \quad \min_{h \in \mathcal{H}_{\mathcal{K}}} \quad \frac{1}{n} \sum_{i=1}^{n} \| h(x_i) - z_i \|_{\mathcal{Z},\epsilon} + \frac{\Lambda}{2} \| h \|_{\mathcal{H}_{\mathcal{K}}}^2 ,
\end{aligned}
$$

are given by eq:expansion, with $\hat{\Omega} = \hat{W} V^{-1}$, and $\hat{W}$ the solution to the respective finite dimensional dual problems

Specific dual problems for our losses 2

For the ϵ-ridge, ϵ-SVR and κ-Huber, it holds $\hat{\Omega} = \hat{W} V^{-1}$, with $\hat{W}$ the solution to these finite dimensional dual problems:

$$(D1) \quad \min_{W \in \mathbb{R}^{n \times n}} \quad \frac{1}{2} \left\| AW - B \right\|_{\text{Fro}}^2 + \epsilon \left\| W \right\|_{2,1},$$

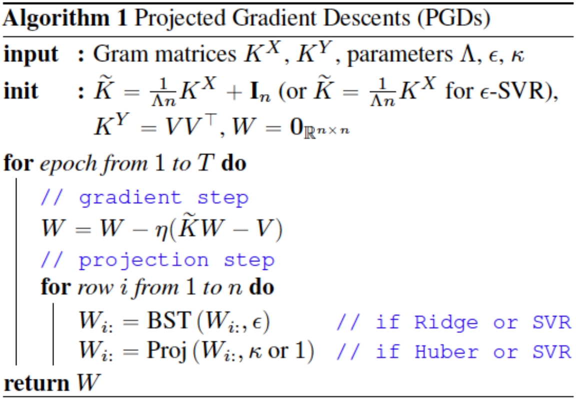
$$\begin{aligned} (D2) \quad & \min_{W \in \mathbb{R}^{n \times n}} \quad \frac{1}{2} \left\| AW - B \right\|_{\text{Fro}}^2 + \epsilon \left\| W \right\|_{2,1}, \\ & \text{s.t.} \quad \left\| W \right\|_{2,\infty} \leq 1, \end{aligned}$$

$$\begin{aligned} (D3) \quad & \min_{W \in \mathbb{R}^{n \times n}} \quad \frac{1}{2} \left\| AW - B \right\|_{\text{Fro}}^2 , \\ & \text{s.t.} \quad \left\| W \right\|_{2,\infty} \leq \kappa, \end{aligned}$$

with V, A, B such that: $VV^\top = K^Y$, $A^\top A = K^X/(\lambda n) + \mathbf{I}_n$ (or $A^\top A = K^X/(\lambda n)$ for the ϵ-SVR), and $A^\top B = V$.

Projected Gradient Algorithm

Projected Gradient Descent algorithms with appropriate projection operator. For instance, (D1) is a multi-task lasso problem (See Obozinski et al. 2010)



Algorithm 1 Projected Gradient Descents (PGDs)

```
input : Gram matrices $K^X$, $K^Y$, parameters $\Lambda$, $\epsilon$, $\kappa$
init  : $\widetilde{K} = \frac{1}{\Lambda n} K^X + \mathbf{I}_n$ (or $\widetilde{K} = \frac{1}{\Lambda n} K^X$ for $\epsilon$-SVR),
        $K^Y = VV^\top$, $W = \mathbf{0}_{\mathbb{R}^{n \times n}}$
for epoch from 1 to $T$ do
    // gradient step
    $W = W - \eta(\widetilde{K}W - V)$
    // projection step
    for row $i$ from 1 to $n$ do
        $W_{i:} = \mathrm{BST}(W_{i:}, \epsilon)$            // if Ridge or SVR
        $W_{i:} = \mathrm{Proj}(W_{i:}, \kappa \text{ or } 1)$   // if Huber or SVR
return $W$
```

Proximal operators

Block Soft Thresholding operator: $\mathrm{BST}(x, \tau) = \left(1 - \tau / \|x\|\right)_+ x$.
Projection operator for (D2) such that $\mathrm{Proj}(x, \tau) = \min\left(\tau / \|x\|, 1\right) x$.

More on IOKR

- Generalization bounds in the context of algorithm stability (extension of Elisseff, 2002; Audiffren and Kadri (2013); Laforgue et al. 2020)
- Deep IOKR: the example of KAE, kernel autoencoder (Laforgue et al. 2019), Deep structured prediction (El Ahmad et al., current work)
- Reduced-rank IOKR (a low-rank approach to IOKR-ridge with excess risk bounds, Brogat-Motte et al. submitted in 2021)

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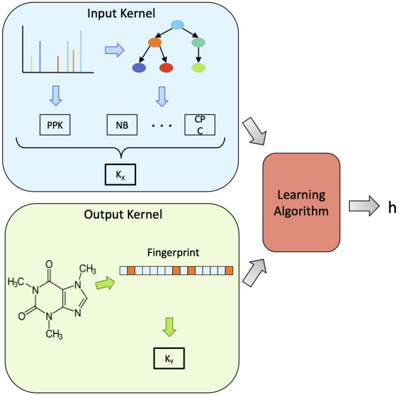
IOKR: the big picture

At training time



Input Kernel

PPK

NB

. . .

CP
C

K_x

Output Kernel

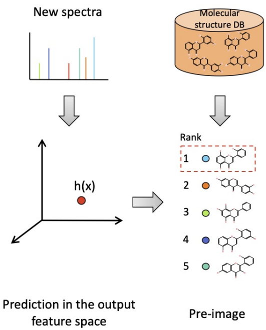
Fingerprint

K_y

Learning Algorithm

h

At testing time



New spectra

Molecular structure DB

h(x)

Rank

1

2

3

4

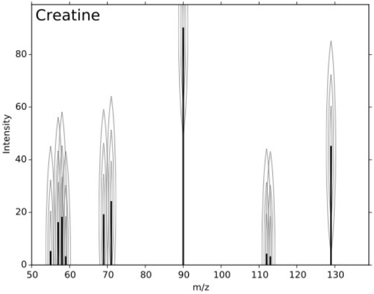
5

Prediction in the output feature space

Pre-image

Input kernels: probability product kernel

- A mass spectrum is defined as a set of peaks: $x = \{x(\ell)\}_{\ell=1}^{n_x}$.
- Each peak is modeled as a 2D normal distribution centered around the observed position: $p_{x(\ell)} \sim \mathcal{N}(x(\ell), \Sigma)$.
- The covariance is shared with all peaks: $\Sigma = \begin{bmatrix} \sigma_m^2 & 0 \\ 0 & \sigma_i^2 \end{bmatrix}$.



Input kernel: probability product kernel

- A spectrum is represented as a mixture of its peak distributions:

$$p_x = \frac{1}{n_x} \sum_{\ell=1}^{n_x} p_{x(\ell)}.$$

- Probability product kernel [Jebara et al., 2004] between the peaks of two spectra x and x':

$$\begin{aligned} k(x, x') &= \int_{\mathbb{R}^2} p_x(\mathbf{z}) p_{x'}(\mathbf{z}) d\mathbf{z} \\ &= \frac{1}{n_x n_{x'}} \frac{1}{4\pi\sigma_m\sigma_i} \sum_{\ell,\ell'=1}^{n_x, n_{x'}} \exp\left(-\frac{1}{4}\left(x(\ell) - x'(\ell')\right)^T \Sigma^{-1} \left(x(\ell) - x'(\ell')\right)\right) \end{aligned}$$

IOKR on metabolite prediction

Metabolite dataset: initially represented by 4136-size fingerprints (Brouard et al., 2016). Tanimoto kernel. Training data: 5579 molecules, Test data: 1359 molecules.

Table 1: Top 1 / 10 / 20 test accuracies (%)

λ	1*e*-6	1*e*-4
RIDGE-IOKR	35.7 | 79.9 | 86.6	38.1 | 82.0 | 88.9
HUBER-IOKR	**38.3** | **82.2** | **89.1**	37.7 | 81.9 | 88.8
ϵ-2-IOKR	37.1 | 81.7 | 88.3	36.3 | 81.2 | 87.9

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Other "complex" output variables

All these problems can be addressed by learning **functions with outputs in a Hilbert space**

Discrete structures

Label Ranking
Sequence, tree prediction
Graph prediction

Multiple Tasks

Hierarchical Classification
Multi-label Classification
Multiple Output Regression

Functions

Infimum of Tasks Learning
Functional Regression
Meta-modeling

Functional Output Regression beyond functional squared loss

Assume now we want to learn to predict a function $h : \mathcal{X} \to \mathcal{Z}$ with functional outputs in $\mathcal{Z} := L_2[\Theta, \mu]$ where Θ is a compact.
Consider again losses that are robust to functional outliers or data-sparse, going beyond the squared loss case solved by Kadri et al.(2016)

The problem then differs from structured prediction

- we need more than just being able to compute inner products between $\psi(y_i)$ and $h(x)$: we need to compute $h(x)(\theta)$
- In the dual problem, the α_is are functions of θ
- A re-parametrization using (linear) splines or truncated approximated eigenspectrum (Lambert et al. 2022) can be used to effectively solve the dual problem for separable OVK of the form: $K(x, x') = k(x, x') T_{k_\Theta} \in \mathcal{L}(L_2(\Theta, \mu))$,
- with $(T_{k_\Theta} f)(\theta) = \int k(\theta, \theta') f(\theta) d\mu(\theta')$.

Conclusion

- The kernel trick used in the output space
- Leveraging vv-RKHS for learning output in infinite dimensional embedding space
- Practical algorithms even for losses more involved than the squared loss
- Other results: generalization bounds within the algorithm stability context

Perspectives

- Scaling up the approaches:
  - Exploit approximations (Random Fourier features: Brault et al. 2017; Projection Learning: Bouche et al. 2020, Sketching, current work of El Ahmad et al.)
- Kernel Learning:
  - Exploiting approximations for both input and output kernel
  - Deep hybrid architecture (learning $\mathcal{K}$) - see for instance (Laforgue et al. 2019, Giffon et al. 2019, Li et al. 2019, Lambert 2021)

Additional thoughts ...

- Handling the output features is not exclusive of kernel methods: see label embedding in one-shot/few-shot learning (Lampert et al. 2015, Djerrab et al. 2018), work of Lerouge et al. (2015) around IODA and Belharbi et al. (2017), for neural networks.
- Leveraging other distances like those in Optimal Transport (see Luise, Rudi et al. 2018) yields to other non-parametric models: see Brogat-Motte et al. 's work on graph prediction with Fused-Gromov-Wasserstein barycenters (ICML 2022).

Codes

- Dualization and Robust losses (`https://github.com/plaforgue/dual_exp`), Pierre Laforgue
- Infinite task Learning: torch-itl (`https://github.com/allambert/torch_itl`), Alex Lambert, Sanjeel Parekh, Dimitri Bouche.
- Reduced-Rank IOKR (not yet public, Luc Brogat-Motte)
- Operalib (`https://github.com/operalib/operalib`) (Romain Brault) RFF for OVK, KRR, IOKR, ITL
- Currently tested : release of a general scikit-learn compatible library with **Hi!Paris** engineering group: if interested to test it, please send me an email.

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References i

- Álvarez, M. A. and Rosasco, L. and Lawrence, N. D., Kernels for vector-valued functions: a review, Foundations and Trends in Machine Learning, 4:3,2012.
- R. Brault, M. Heinonen, F. d'Alché-Buc, Random Fourier Features for Operator-valued Kernels, ACML 2016, (2016)
- R. Brault, A. Lambert, Z. Szabó, M. Sangnier, F. d'Alché-Buc: Infinite Task Learning in RKHSs. AISTATS 2019: 1294-1302
- L. Brogat-Motte, R. Flamary, F. d'Alché-Buc:Learning to Predict Graphs with Fused Gromov-Wasserstein Barycenters. To appear in Proc. of ICML 2022.
- C. Brouard, F. d'Alché-Buc, M. Szafranski: Semi-supervised Penalized Output Kernel Regression for Link Prediction. ICML 2011: 593-600.
- C. Brouard, F. d'Alché-Buc, M. Szafranski, Input Output Kernel Regression for supervised and semi-supervised structured output learning, JMLR, oct. 2016
- C. Brouard, H. Shen, K. Dührkop, F. d'Alché-Buc, S. Böcker, J. Rousu: Fast metabolite identification with Input Output Kernel Regression. Bioinformatics 32(12): 28-36 (2016)
- C. Ciliberto, L. Rosasco, A. Rudi: A General Framework for Consistent Structured Prediction with Implicit Loss Embeddings. J. Mach. Learn. Res. 21: 98:1-98:67 (2020)
- C. Ciliberto, L. Rosasco, A. Rudi: A Consistent Regularization Approach for Structured Prediction. NIPS 2016: 4412-4420

References ii

- T. Evgeniou, C. A. Micchelli, M. Pontil: Learning Multiple Tasks with Kernel Methods. Journal of Machine Learning Research 6: 615-637 (2005)
- P. Geurts, L. Wehenkel, F. d'Alché-Buc: Kernelizing the output of tree-based methods. ICML 2006: 345-352
- Pierre Geurts, Nizar Touleimat, Marie Dutreix, F. d'Alché-Buc: Inferring biological networks with output kernel trees. BMC Bioinform. 8(S-2) (2007)
- Pierre Geurts, Louis Wehenkel, Florence d'Alché-Buc: Gradient boosting for kernelized output spaces. ICML 2007: 289-296
- Luc Giffon, Stéphane Ayache, Thierry Artières, Hachem Kadri: Deep Networks with Adaptive Nyström Approximation. IJCNN 2019: 1-8
- Hachem Kadri, Emmanuel Duflos, Philippe Preux, Stéphane Canu, Manuel Davy: Nonlinear functional regression: a functional RKHS approach. AISTATS 2010: 374-380
- Hachem Kadri, Mohammad Ghavamzadeh, Philippe Preux: A Generalized Kernel Approach to Structured Output Learning. ICML (1) 2013: 471-479
- Hachem Kadri, Emmanuel Duflos, Philippe Preux, Stéphane Canu, Alain Rakotomamonjy, Julien Audiffren: Operator-valued Kernels for Learning from Functional Response Data. J. Mach. Learn. Res. 17: 20:1-20:54 (2016)
- A. Lambert, D. Bouche, Z. Szabo, F. d'Alché-Buc:Functional Output Regression with Infimal Convolution: Exploring the Huber and ϵ-insensitive loss. To appear in Proc. of ICML 2022.

References iii

- P. Laforgue, S.Clémencon, Florence d'Alché-Buc: Autoencoding any Data through Kernel Autoencoders. AISTATS 2019: 1061-1069
- P. Laforgue, A. Lambert, L. Brogat-Motte, F. d'Alché-Buc: Duality in RKHSs with Infinite Dimensional Outputs: application to robustness, ICML 2020.
- Micchelli, C. A. and Pontil, M. and Ying, Y., Universal MultiTask Kernels, Journal of Machine Learning Research, 9,2008.
- M. Moussab, A. Garcia, M. Sangnier, F. d'Alché-Buc, Output Fisher Embedding Regression, Machine Learning Journal, (2018)
- M. Sangnier, O. Fercoq, F. d'Alché-Buc, Joint quantile regression in vector-valued RKHSs, NIPS 2016:3693-3701, (2016)
- M. Sangnier, O. Fercoq, F. d'Alché-Buc, Data sparse nonparametric regression with ϵ-insensitive losses, ACML 2017, (2017)
- L. Wenliang, D. J. Sutherland, H. Strathmann, A. Gretton. Proceedings of the 36th International Conference on Machine Learning (ICML), PMLR 97:6737-6746

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More about regularized least-squares regression: a reduced rank approach

Let $\lambda_1, \lambda_2 > 0$ and $p \in \mathbb{N}^*$. Let $\mathcal{P}_p$ be the set of the orthogonal projections from $\mathcal{Z}$ to $\mathcal{Z}$ of rank p.

We consider the estimator $x \to P\hat{h}_{\lambda_2}(x)$ where P is defined as

$$P := \text{argmin }_{P \in \mathcal{P}_p} \mathbb{E}[\|Ph^*(x) - h^*(x)\|_{\mathcal{Z}}^2]. \tag{7}$$

Nevertheless, P is unknown, thus we estimate it with $\hat{P}$ defined by

$$\hat{P} := \text{argmin }_{P \in \mathcal{P}_p} \frac{1}{n} \sum_{i=1}^{n} \|P\hat{h}_{\lambda_1}(x_i) - \hat{h}_{\lambda_1}(x_i)\|_{\mathcal{Z}}^2. \tag{8}$$

and we propose the estimator

$$\boxed{\hat{h}_{\lambda_1,\lambda_2,p}(x) = \hat{P}\hat{h}_{\lambda_2}(x)} \tag{9}$$

Reduced-rank regression in Structured Prediction

Novel estimator for IOKR in structured prediction ($\mathcal{Z} := \mathcal{H}_{k_\mathcal{Y}}$)

$$\hat{f}(x) = \text{argmin }_{y \in \mathcal{Y}} \|\hat{P}\hat{h}(x) - \psi(y)\|^2_{\mathcal{Z}}. \tag{10}$$

Algorithm	ridge-IOKR	Reduced-rank IOKR
Training	$\mathcal{O}(n^3)$	$\mathcal{O}(2n^3)$
Decoding	$\mathcal{O}(n_{test}\, n\lvert\mathcal{Y}\rvert)$	$\mathcal{O}(n_{test}\, p\lvert\mathcal{Y}\rvert)$

Table 2: Time complexity of IOKR versus reduced-rank IOKR.

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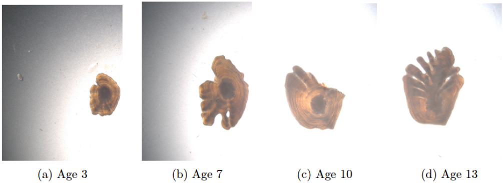
Fish age prediction

Goal: monitoring fish resources with the goal of avoiding overfishing. Fishes are collected and their age is measured from images of their otoliths (part of inner ear).

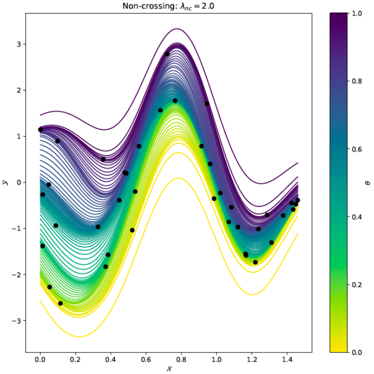

(a) Age 3 (b) Age 7 (c) Age 10 (d) Age 13

Fish bone images of otoliths at different ages. Dataset from the Norwegian Marine Data Center (Ordonez et al. 2020).

Question: automatic prediction of fish age from otolith images ?
(Wu et al., Nature (2009), Martino et al. 2019, Ordonez et al. 2020)

Quantile regression for fish age prediction

Many reasons for quantile regression: outliers in the data, more meaningful to get median, quartiles ...



Question: Predict any θ-quantile of Y given x, for $\theta \in (0, 1)$ [Brault et al. 2019]

Multi-task learning and matrix-valued kernels

Multi-task learning problems:

- Generally solved by leveraging the dependency between tasks through some appropriate regularization.
- Matrix-valued kernels is a relevant option for multi-task learning (Michelli and Pontil 2005, Baldassare et al. 2012, Alvarez et al. 2012, Kadri et al. 2013, Lim et al. 2013, Sangnier et al. 2016)

Infinite Task Learning

Jointly solve an infimum of tasks, i.e. tasks parameterized by a continuous parameter

- At training time, more opportunities to jointly control the solution(s)
- At testing time, a new task can be solved

N.B. Parametric task learning first defined by Takeuchi et al. in 2013 was solved for piecewise-linear output functions.

Infinite Task Learning

Framework proposed: learn function-valued functions

$$\text{input} \mapsto (\text{parameter} \mapsto \text{output})$$

$$\mathcal{X} \to (\Theta \to \mathcal{Y})$$

Goal : Learn a global function while preserving desired properties of the output function over the θ space Θ.
Related works: Functional Output Regression problem (see for instance Kadri et al. 2010, Kadri et al. 2016)

The example of quantile regression

- $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ random variables
- $\theta \in (0, 1)$

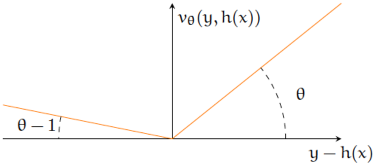
The example of quantile regression

- $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ random variables
- $\theta \in (0, 1)$

Conditional quantile:

$$q(x) = \inf \ \{y \in \mathbb{R} \ , \ P(Y \leq y \mid X = x) = \theta\}$$

The example of quantile regression



Learn a function $h : \mathcal{X} \times \mathcal{H}_{k_\Theta}$ in order to approach the conditional quantile of Y given x for any $\theta \in (0, 1)$.

Pinball loss: $\ell(\theta, y, h(x)(\theta)) = \rho_\theta(y - h(x)(\theta)) =$
$max(\theta(y - h(x)(\theta)), (1 - \theta)(y)h(x)(\theta))$
Why ?
Remind that a quantile $q_Y(\theta) := \arg\min_u \mathbb{E}[\rho_\theta(Y - u)]$.

What loss for infinite task learning?

Remember θ is not observed

$$I_\ell(y, h(x)) = \int_\Theta \ell(\theta, y, h(x)(\theta)) d\mu(\theta)$$

N.B. In the following, $\Theta = \mathbb{R}$ and for μ, the Lebesgue measure.

Task parameterized by a continuous parameter θ

(Regularized) ERM setting
minimize the regularized empirical risk in $\mathcal{H} \subset \mathcal{F}(\mathcal{X}, \mathcal{F}(\Theta, \mathcal{Y})$ using a training set $\mathcal{S}_n = (x_i, y_i)_{i=1}^n \subset (\mathcal{X} \times \mathcal{Y})^n$, $\lambda > 0$ and $\Omega(h)$ is a regularization term.

$$L_n(h) = \frac{1}{n} \sum_{i=1}^{n} l_\ell(y_i, h(x_i)) + \lambda \Omega(h)$$

First question: Sampled Empirical Risk

Estimating the integral using a weighted sample $\{\theta_1, \dots, \theta_m\}$ using quadrature rules, Monte-Carlo or Quasi-Monte-Carlo estimators.
In practice, we take:

$$I_\ell^m = \sum_{j=1}^{m} \eta_j \ell(\theta_j, y, h(x)(\theta_j))$$

Monte-Carlo: $\eta_j = \frac{1}{m}$

Second Question: Hypothesis spaces

Say $h \in \mathcal{H}$ is our predictive model, we need to define:

- output functional space $h(x) \in \mathcal{Z} \subset \mathcal{F}(\Theta, \mathcal{Y})$
- hypothesis space: $h \in \mathcal{H} \subset \mathcal{F}(\mathcal{X}, \mathcal{Z})$

Again vv-RKHS come into play:
Take two scalar kernels $k_{\mathcal{X}} \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $k_{\Theta} \colon \Theta \times \Theta \to \mathbb{R}$, construct

$$\mathcal{K} \colon \begin{cases} \mathcal{X} \times \mathcal{X} & \to \mathcal{L}(\mathcal{H}_{k_\Theta}) \\ (x, z) & \mapsto k_{\mathcal{X}}(x, z) I_{\mathcal{H}_{k_\Theta}} \end{cases}$$

With this choice: structure:$= \mathcal{H}_{\mathcal{K}} \simeq \mathcal{H}_{k_{\mathcal{X}}} \otimes \mathcal{H}_{k_\Theta}$ *i.e*

Infinite task learning

In infinite task learning, we observed $(x_i, y_i)_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n$ and we search for functions:
$\mathcal{X} \rightarrow \mathcal{H}_{k_\Theta}$ where $\mathcal{H}_{k_\Theta}$ is $\mathcal{Y}$-valued.

- Our target problem is learning a function-valued function without observing functional output data during training. We have only surrogate data.
- What is important in practice is to be able to compute during training and testing, the images of functions $\theta \rightarrow h(x)(\theta)$ for any $x \in \mathcal{X}$.

Optimization

Let us take advantage of the regularized sampled risk and exploit a representer theorem.
Then solve in the primal space.

The example of infinite quantile regression

- $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ random variables
- $\theta \in (0, 1)$

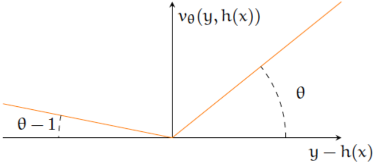
The example of infinite quantile regression

- $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ random variables
- $\theta \in (0, 1)$

Conditional quantile:

$$q(x) = \inf \ \{y \in \mathbb{R} \ , \ P(Y \leq y \mid X = x) = \theta\}$$

The example of infinite quantile regression



Pinball loss:
$\ell(\theta, y, h(x)(\theta) = \rho_\theta(y{-}h(x)(\theta)) = max(\theta(y{-}h(x)(\theta)), (1{-}\theta)(y)h(x)(\theta))$
Why ?
Remind that a quantile $q_Y(\theta) := \arg\min_u \mathbb{E}[\rho_\theta(Y - u)]$.

The example of infinite quantile regression

Properties we want for $\theta \to \hat{q}(x)(\theta)$, given x

- Continuous
- Smooth
- Nondecreasing

This yields i) take Gaussian kernel for k_Θ and ii) appropriate penalty on shape.

Hypothesis of the representer theorem with shape penalty

$$\hat{h} = \arg \min_{h \in \mathcal{H}_{\mathcal{K}}} \frac{1}{n} \sum_{i=1}^{n} l_{\ell}^{m}(h(x_i), y_i) + \frac{\lambda}{2} \|h\|_{\mathcal{H}_{\mathcal{K}}}^{2} + \widetilde{\Omega}_{nc}(h). \qquad (11)$$

with

$$\widetilde{\Omega}_{n,m}(h) := \lambda_{nc} \frac{1}{nm} \sum_{i,j=1}^{n,m} \left| -(\partial_{\Theta} h)(\tilde{x}_i)(\tilde{\theta}_j) \right|_{+}.$$

Representer theorem with shape penalty

Theorem

The previous joint quantile regression problem admits a unique solution $\hat{h} \in \mathcal{H}_{\mathcal{K}}$, and there exist $(\alpha_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n \times m}$ and $(\beta_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n \times m}$ such that for all $(x, \theta) \in \mathcal{X} \times \Theta$,

$$\hat{h}(x)(\theta) = \sum_{i,j}^{n,m} k_{\mathcal{X}}(x, x_i) \left(\alpha_{ij} k_{\Theta}(\theta, \theta_j) + \beta_{ij} \partial_2 k_{\Theta}(\theta, \tilde{\theta}_j) \right). \tag{12}$$

For simplicity, we took the same number m for estimating the non-crossing constraint and the integral loss I_ℓ.

Relevance of shape penalty on a toy problem

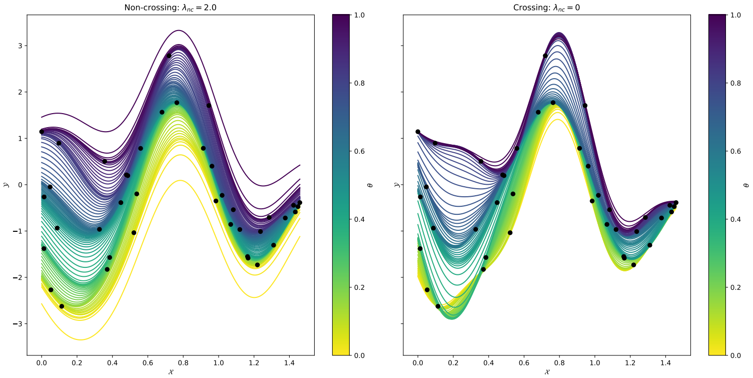


Figure 1: Effect of regularization parameter on infinite quantile regression

Intensive experimental results in (Brault, Lambert, et al. 2019) on a large set of UCI repositories.

Quantile regression on fish age (otolith problem)

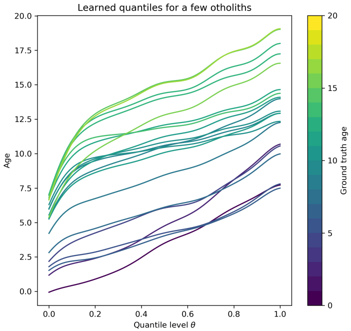


Figure 2: Quantile curve predicted by ITL

Recent results obtained by Lambert with an hybrid architecture: pre-trained convolutional network (Ordoñez et al. 2020) that feeds Random Fourier features for x and θ.

Comparison with structured output regression

$\mathcal{X}$: input space

Structured prediction

$\mathcal{Y}$: finite set of structured objects

$k_{\mathcal{Y}}$: kernel over $\mathcal{Y}$

$\mathcal{Z} := \mathcal{H}_{k_{\mathcal{Y}}}$: RKHS associated to $k_{\mathcal{Y}}$

$$\mathcal{X} \xrightarrow{h} \mathcal{Z} \xrightarrow{d} \mathcal{Y}$$

Goal: obtain $f(x) = d \circ h(x)$

Infinite Task learning

$\mathcal{Y}$: output (observation) space

Θ: task parameter space

k_{Θ}: kernel over Θ

$\mathcal{Z} := \mathcal{H}_{k_{\Theta}}$: RKHS associated to k_{Θ}

$$\mathcal{X} \xrightarrow{h} \underbrace{(\Theta \to \mathcal{Y})}_{\mathcal{Z}}$$

Goal: obtain $h(x)(\theta)$