Gaussian Process Based Reachability Analysis

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Problem
For dynamic systems, reachability analysis computes the set of all possible trajectories of the states that a dynamical system can reach, under model uncertainties and perturbations, starting from a set of initial conditions [2]. Reachability analysis helps to determine: 1) whether the states of the dynamical system ever can belong to a set characterised as unsafe in finite or infinite time; 2) whether the trajectories of the dynamical system can reach a goal region around a set point; and 3) the parameter sets for the dynamical system which satisfy given requirements, among others.

Basic Concepts
Given a dynamical system \( \dot{x} = f(x(t), u(t), \delta(t)) \), where \( t \) is the time, \( x \) is the state, \( u \) is the input, and \( \delta \) is the model uncertainties, the reachable set at a certain instant of time \( T \) is

\[
\mathcal{R}(T) = \left\{ x(T) = \int_0^T f(x(\tau), u(\tau), \delta(\tau)) \, d\tau \mid x(0) \in \mathcal{X}_0, \ u \in \mathcal{U}, \ \delta \in \Delta \right\},
\]

where \( \mathcal{X}_0, \mathcal{U}, \Delta \) are the bounded sets for possible initial states, the control input and model uncertainties.

Methods
Literature
1) Set-propagation techniques that depend on ellipsoids, Taylor series method, zonotopes, level sets and intervals and simulation-based techniques that depends on numerical solution to ODEs are popular approaches to determine reachable sets.
2) Methods making use of support functions have also been investi- gated.

Proposed in This Work
A Gaussian process (GP) based approach to compute the reachability set. Gaussian process [3] is a nonparametric stochastic process over functions, it infersences highly nonlinear latent function from (noisy) observations in a Bayesian way. In the proposed approach, the finite time simulation of the dynamical model will be regarded as a ‘black-box’ which maps certain set of inputs (including initial conditions, model uncertainties and control input) to the output which is the set of states at a specific time.

Dynamic Model
The performance of the proposed algorithm is tested with an abstract, stable linear time-invariant five-dimension system used in [2],

\[
\dot{x} = Ax + u(t)
\]

\[
A = \begin{bmatrix}
-1 & -4 & 0 & 0 & 0 \\
-4 & -1 & 0 & 0 & 0 \\
0 & 0 & -3 & 1 & 0 \\
0 & 0 & -1 & -3 & 0 \\
0 & 0 & 0 & 0 & -2
\end{bmatrix}
\]

The initial set is chosen to be \( \mathcal{X}_0 = \{x_i, i = 1, \ldots, 5 \mid x_i \in [0.9, 1.1]\} \), control input set is \( \mathcal{U} = \{u_i, i = 1, \ldots, 5 \mid u_i \in [9.98, 10.02]\} \), and there is no model uncertainties. We compute the reachable sets when \( T = 0.05s, T = 0.10s \) and \( T = 0.1s5 \).

The GP model is constructed from a training dataset \( \mathcal{L} \) which consists of \( N \) samples of \( \theta = (x(0), u, \delta) \) and their observations \( y(\theta) \), for a given \( T \). The training dataset \( \mathcal{L} = \{D, y\} \), where \( D = \{\theta_1, \theta_2, \ldots, \theta_N\}, \ y = \{y(\theta_1), y(\theta_2), \ldots, y(\theta_N)\} \). A Gaussian posterior predictive distribution at any test \( \theta^* \) is specified by mean \( \mu(\theta^*) \) and covariance \( \Sigma(\theta^*) \).

Results and Future Work
Results
We obtain reachability set results from three existing available methods, LTI-Reachability Algorithm, LTI-Constrained-ReachabilityAlgorithm and LTI-Side-Info-Reachability Algorithm, as proposed in [1], are given here for comparison.

![Image 1](image1.png)
![Image 2](image2.png)

Figure 1. Projections of the reachable sets on \( x_3-x_4 \) plane(left) and \( x_2-x_4 \) plane(right)

Figures 1 gives the projections of the reachable sets of the LTI system on \( x_1-x_2 \) and \( x_3-x_4 \) state plane respectively. The reachable sets are computed via LTI-Reachability Algorithm(\( \mathcal{R}_A \)), LTI-Constrained-ReachabilityAlgorithm(\( \mathcal{R}_A \)), LTI-Side-Info-Reachability Algorithm(\( \mathcal{R}_A^s \)) and GP based algorithm (black dashed line) from noisy input-state data. The rectangles in the left bottom of Figures 1 indicate the initial set, and then from the bottom to the top, three groups of reachable sets are for \( T = 0.05, T = 0.10 \) and \( T = 0.15, \) respectively. From both projections we can see that the algorithms provided in [1] over-approximate the reachable sets, and the Gaussian process based algorithm proposed in this work slightly under-approximates the actual reachable sets given in blue(\( \mathcal{R}_G \)). The execute time of those three algorithms from [1] is 1.8142 minutes in total, and the execute time of GP based algorithm is 0.5724 minutes (less than 1.8142/3 minutes).

Future Work
In this work, the five dimension system states are modelled by five independent GPs. Future work will be using the multiple output Gaussian process rather than the single GP, in order to improve the model accuracy and reduce the computational complexity and employ the approach to complex aerospace simulation models.

References

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