Supervised deep learning for stochastic lid-driven cavity flows

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Steady lid-driven cavity flow

Consider an incompressible fluid in a two-dimensional square cavity, that is evenly pushed in one direction at the lid. The described problem is a well-known benchmark problem in fluid dynamics and it is modeled with the **Navier-Stokes equations** (NSE). We consider an equivalent formulation of the NSE, called **vorticity-streamfunction formulation**.

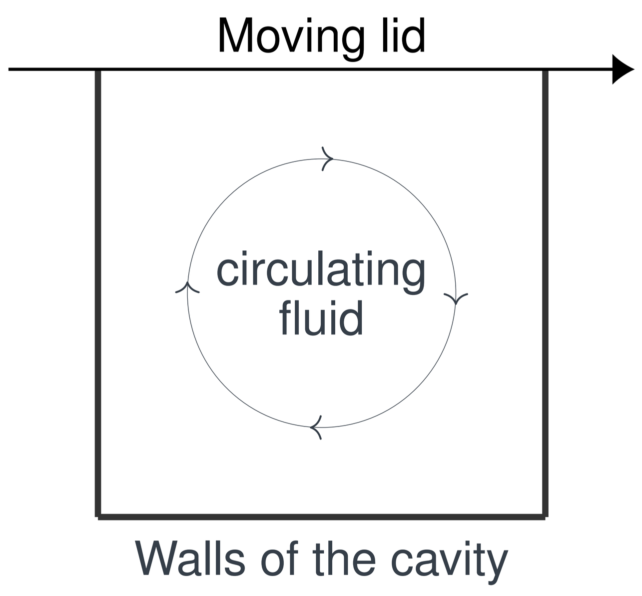

Figure 1: Lid-driven cavity flow.

Vorticity-streamfunction formulation of the NSE

Let $\vec{v} : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ be a differentiable flow velocity of an incompressible fluid. The **vorticity** $\omega : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ is defined as the curl of the flow velocity, $\omega = \nabla \times \vec{v}$, and the **streamfunction** $\psi : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ is a function satisfying the following equations:

$$\vec{v}_1 = \frac{\partial \psi}{\partial x_2}, \quad \vec{v}_2 = -\frac{\partial \psi}{\partial x_1}.$$

The vorticity-streamfunction formulation of the Navier-Stokes equations reads:

$$-\Delta \psi = \omega,$$

$$\frac{\partial \omega}{\partial t} - \Delta \omega = \mathrm{Re}\left(\frac{\partial \psi}{\partial x_1}\frac{\partial \omega}{\partial x_2} - \frac{\partial \psi}{\partial x_2}\frac{\partial \omega}{\partial x_1}\right),$$

where Re denotes the Reynolds number.

Problem setup

We impose no-slip conditions on the impenetrable walls of the cavity and obtain the following boundary conditions:

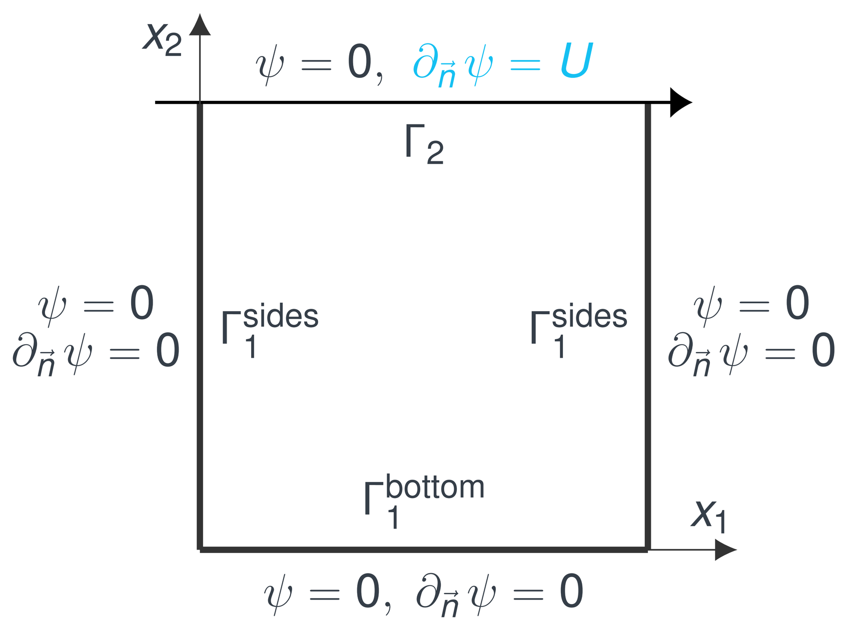

Figure 2: Boundary conditions of lid-driven cavity flow.

Stochastic lid-driven cavity flow

Let the driving velocity U be of stochastic nature, namely $U \sim \mathcal{U}([0.25, 5])$. A solution to the stochastic problem is understood in the **sense of Foiaş-Prodi**.

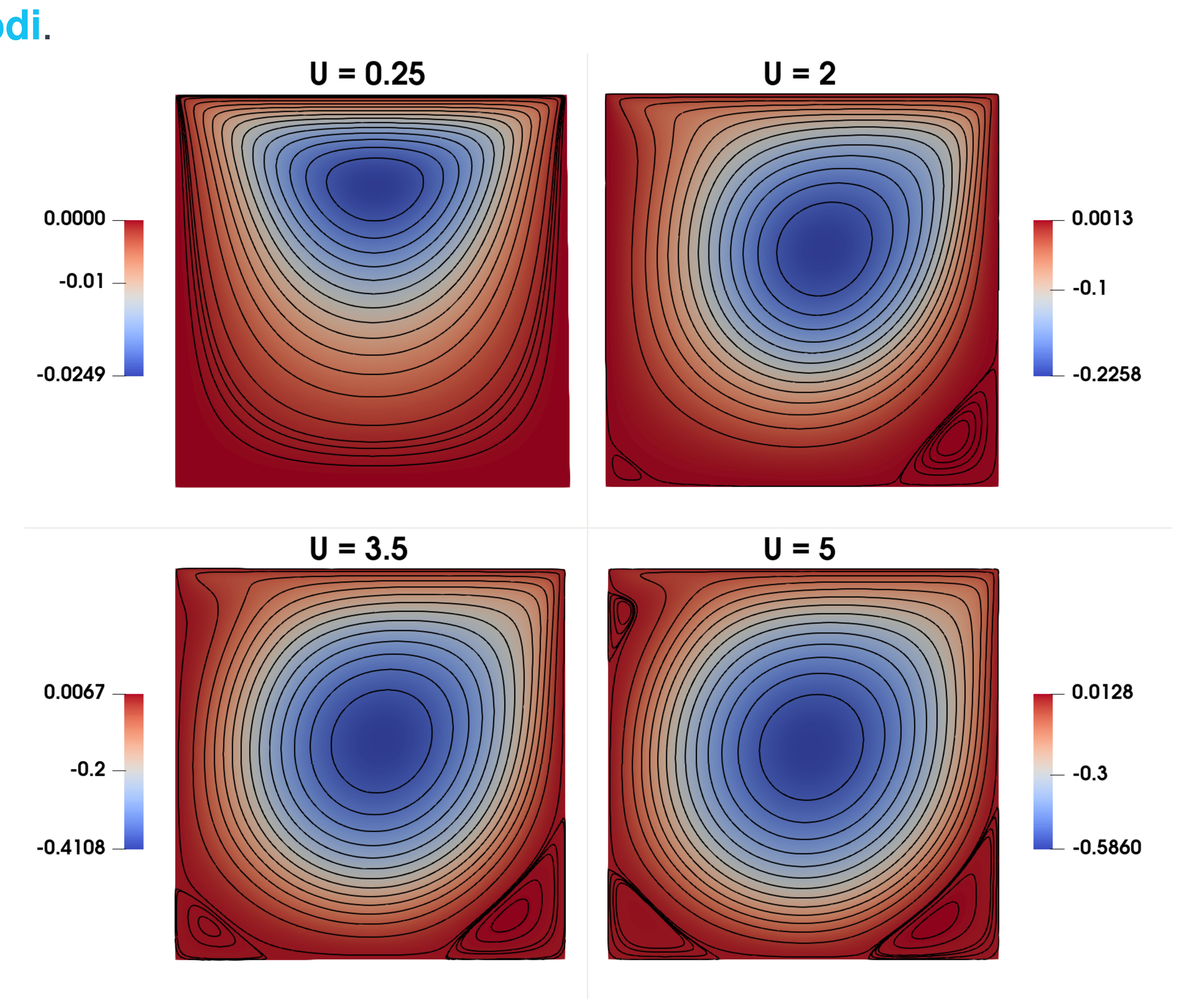

Figure 3: Streamfunction contours for different driving velocities.

Supervised deep learning approach

In a stochastic setup like this, it is crucial to have access to many pathwise samples of the experiment in order to **quantify uncertainties**, and to gain a better understanding of the underlying stochastic nature. Solving the lid-driven cavity flow with classical methods is of immense computational expense → **machine learning assisted approach**.

StreamNet

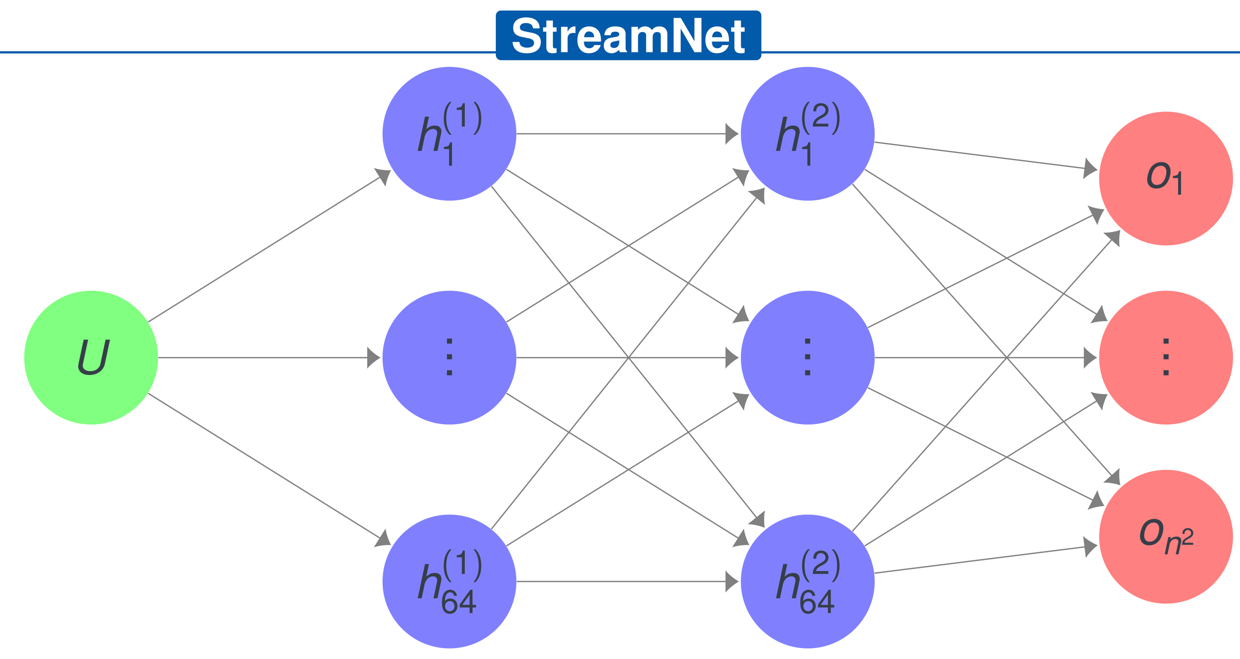

Figure 4: Architecture of StreamNet

- The neural network takes a **driving velocity U as input**, and produces an **approximated streamfunction** on a $n \times n$ grid as output.
- The network is trained on **data sets of different sizes**, containing $k \in \{10, 25, 50, 100, 250, 500\}$ in [0.25, 5] equidistant points, representing driving velocities with corresponding streamfunctions produced by a FD-solver.
- StreamNet is used for computationally fast **Monte Carlo estimates**.
- The neural network is able to reliably predict streamfunctions for **unseen data exceeding the domain of the training data** ($U = 5 \to U = 7.5$, MSE $\sim 10^{-4}$).
- StreamNet captures the secondary vortices quite well, even though they are represented by very small streamfunction values.

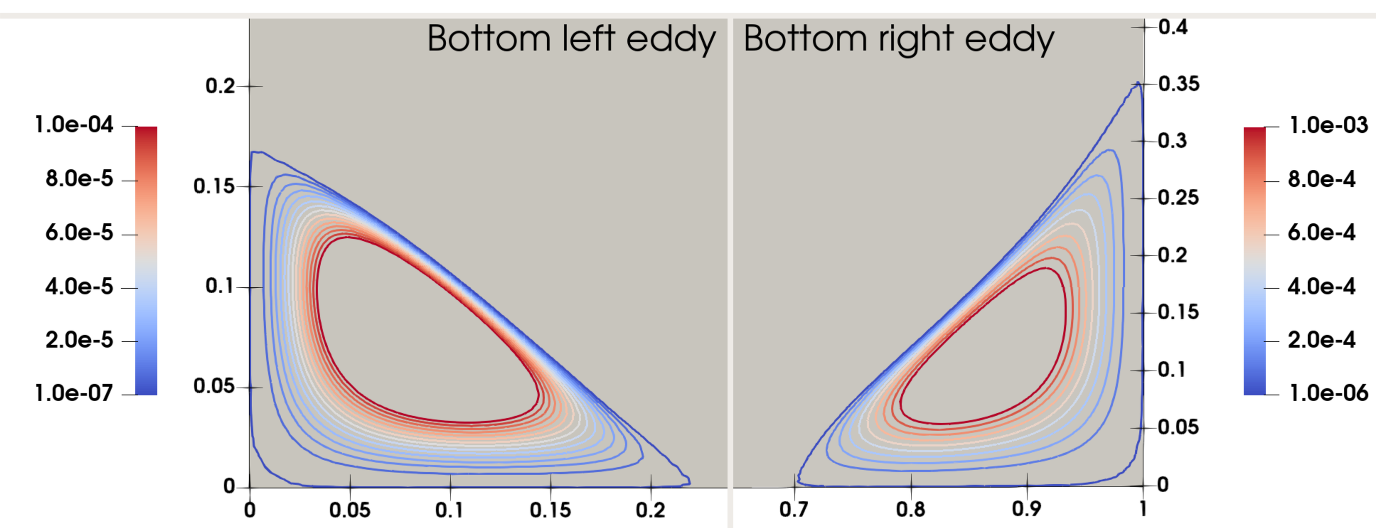

Figure 5: Streamfunction values of secondary vortices.

# training points	MSE	MAE	MaxAE
500	$9.39 \cdot 10^{-8}$	$1.96 \cdot 10^{-4}$	$1.74 \cdot 10^{-3}$
250	$1.02 \cdot 10^{-7}$	$2.03 \cdot 10^{-4}$	$1.49 \cdot 10^{-3}$
100	$1.30 \cdot 10^{-7}$	$2.40 \cdot 10^{-4}$	$1.60 \cdot 10^{-3}$
50	$1.55 \cdot 10^{-7}$	$2.81 \cdot 10^{-4}$	$1.77 \cdot 10^{-3}$
25	$4.38 \cdot 10^{-7}$	$4.72 \cdot 10^{-4}$	$2.58 \cdot 10^{-3}$
10	$2.74 \cdot 10^{-6}$	$1.16 \cdot 10^{-3}$	$5.41 \cdot 10^{-3}$
Benchmark: untrained network	0.029	0.114	0.361

Table 1: Error metrics for StreamNet evaluated on 500 data points.

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