Supervised deep learning for stochastic lid-driven cavity flows

Steady lid-driven cavity flow

Consider an incompressible fluid in a two-dimensional square cavity, that is evenly pushed in one direction at the lid. The described problem is a well-known benchmark problem in fluid dynamics and it is modeled with the Navier-Stokes equations (NSE). We consider an equivalent formulation of the NSE, called vorticity-streamfunction formulation.

Vorticity-streamfunction formulation of the NSE

Let \( \mathbf{v} : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R} \) be a differentiable flow velocity of an incompressible fluid. The vorticity \( \omega = \nabla \times \mathbf{v} \), and the streamfunction \( \psi : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R} \) is a function satisfying the following equations:

\[
\begin{align*}
-\Delta \psi &= \omega, \\
\frac{\partial \omega}{\partial t} - \Delta \omega &= \text{Re} \left( \frac{\partial \psi}{\partial x_1} \frac{\partial \omega}{\partial x_2} - \frac{\partial \psi}{\partial x_2} \frac{\partial \omega}{\partial x_1} \right),
\end{align*}
\]

where \( \text{Re} \) denotes the Reynolds number.

We impose no-slip conditions on the impenetrable walls of the cavity and obtain the following boundary conditions:

\[
\begin{align*}
\psi &= 0, \quad \partial_n \psi = U \\
\frac{\partial \psi}{\partial n} &= 0 \\
\psi &= 0 \\
\frac{\partial \psi}{\partial n} &= 0
\end{align*}
\]

Stochastic lid-driven cavity flow

Let the driving velocity \( U \) be of stochastic nature, namely \( U \sim \mathcal{U}([0, 25], 5] \). A solution to the stochastic problem is understood in the sense of Foias-Prodi.

Supervised deep learning approach

In a stochastic setup like this, it is crucial to have access to many pathwise samples of the experiment in order to quantify uncertainties, and to gain a better understanding of the underlying stochastic nature. Solving the lid-driven cavity flow with classical methods is of immense computational expense → machine learning assisted approach.

Figure 1: Lid-driven cavity flow.

Problem setup

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\end{align*}
\]

Figure 2: Boundary conditions of lid-driven cavity flow.

Stochastic lid-driven cavity flow

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Figure 3: Streamfunction contours for different driving velocities.

Table 1: Error metrics for StreamNet evaluated on 500 data points.

<table>
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<th># training points</th>
<th>MSE</th>
<th>MAE</th>
<th>MaxAE</th>
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References


