Kernel-based probabilistic integration

1. Select integration nodes \( \Rightarrow \mathbb{E}[g(X)] \approx \frac{1}{N} \sum_{i=1}^{N} g(x_i) \)

**Candidate set**: \( S \) is a fairly dense finite subset of \( \mathbb{R}^2 \) with size \( N \gg n \) that emulates the target distribution (e.g., a large Sobol' sequence, available data as in Fig. 2).

**Kernel herding criterion** \([4, 6]\): at iteration \( n + 1 \), for a given design of experiments (DoE) \( X_n = \{x_1, \ldots, x_n\} \), and a given kernel \( k \)

\[
x^{(n+1)} = \arg \min_{x \in S} \frac{1}{n} \sum_{i=1}^{n} k(x, x_i) - \frac{1}{N} \sum_{i=1}^{N} k(x, x_i)
\]

2. Compute optimal weights for integration \( \Rightarrow \mathbb{E}[g(X)] \approx \sum_{i=1}^{N} w_i g(x_i) \)

**Optimal weights for quadrature** \([5]\): for a given DoE \( X_n \) and a given kernel \( k \)

\[
w^* = P(X_n|K_{n-1})^{-1}
\]

with potentials \( P(X_n) = \{ f \int k(x, x_i) f(x) \, dx, \ldots, f \int k(x, x_N) f(x) \, dx \} \) and variance-covariance matrix \( \{ K_{n-1} \}_{ij} = k(x_i, x_j) \)

**Sampling a bivariate random mixture**

**Conclusions & Perspectives**

- Combining kernel herding with optimal weights is an efficient integration method
- This method is sensitive to the chosen kernel and its hyper-parameters
- This method allows direct sampling from available empirical distribution
- Active learning methods might be more efficient but are harder to distribute (HPC)
- Determine influential system variables using advanced sensitivity analysis methods
- Adapt kernel-based sampling methods to reliability analysis problems

**References**