Improved cross entropy method with Bernoulli mixture model

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Introduction

Background

- Infrastructure networks, such as power grids and water supply systems are essential for societies.
- The quantification of the reliability, or conversely, the probability of failure of such systems under hazards is crucial for managing their reliability.

Aim of this work

- Developing simulation based method for static network reliability, with a particular focus on rare event estimation.

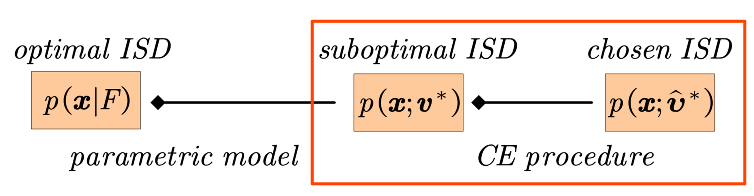
Cross entropy method (Rubinstein, 1999)

- The basic idea is to choose the importance sampling distribution (ISD) through minimizing the Kullback–Leibler (KL) divergence between the optimal ISD, $p(x|F)$ and a chosen parametric model $p(x;v)$.
- The optimal parameter can be calculated through

$$\boldsymbol{v}^* = \underset{\boldsymbol{v}\in\mathcal{V}}{\operatorname{argmin}}\; D_{\mathrm{KL}}(p(\boldsymbol{x}|F) \rightharpoonup p(\boldsymbol{x};\boldsymbol{v})) = \underset{\boldsymbol{v}\in\mathcal{V}}{\operatorname{argmax}} \int_{\Omega} \mathbb{I}\{g(\boldsymbol{x}) \le 0\}\, p(\boldsymbol{x}) \ln(p(\boldsymbol{x};\boldsymbol{v}))\, d\boldsymbol{x}$$

- In practice we maximize another objective function

$$\hat{\boldsymbol{v}}^* = \underset{\boldsymbol{v}\in\mathcal{V}}{\operatorname{argmax}} \frac{1}{N_s} \sum_i \frac{\mathbb{I}\{g(\boldsymbol{x}_i) \le 0\}\, p(\boldsymbol{x}_i)}{p_{ref}(\boldsymbol{x}_i)} \ln(p(\boldsymbol{x}_i;\boldsymbol{v})) \qquad \boldsymbol{x}_i \sim p_{ref}(\cdot)$$

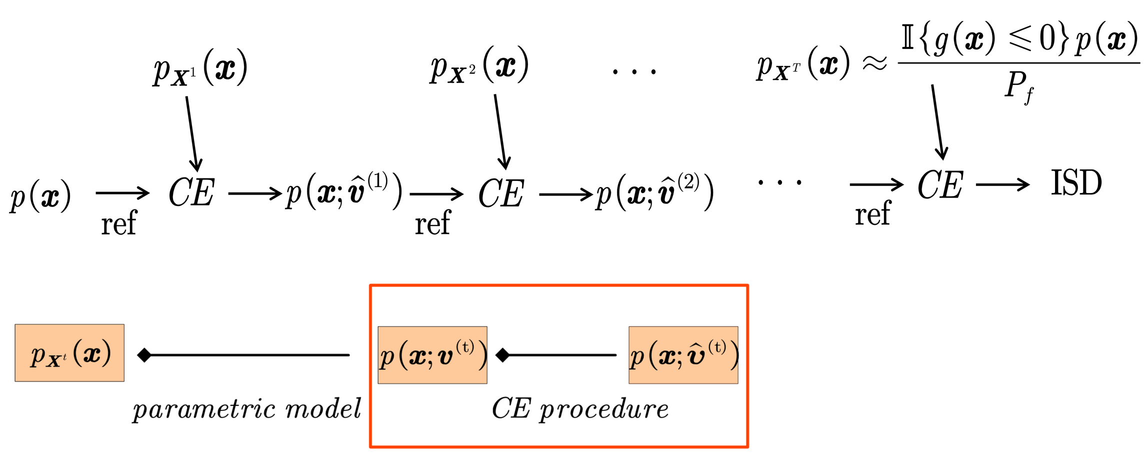


Cross entropy method for rare event estimation

- Define a sequence of intermediate target distributions connecting the input distribution to the optimal ISD.

$$p_{X^t}(\boldsymbol{x}) = \frac{1}{Z_t} p(\boldsymbol{x}) \mathbb{I}\{g(\boldsymbol{x}) \le \gamma_t\},\ t = 0, \ldots, T,\ \infty = \gamma_0 > \gamma_1 > \ldots > \gamma_T \approx 0$$

- The CE optimization problem is then solved iteratively to get a good ISD.
- $p(x;\hat{v}^{(t-1)})$ is chosen as the reference distribution for the CE procedure at level t.



Improved cross entropy method (Papaioannou et.al. 2019)

- The sequence of intermediate target distributions is defined by a smooth transition of the indicator function

$$p_{X^t}(\boldsymbol{x}) = \frac{1}{Z_t} p(\boldsymbol{x}) \Phi\left(-\frac{g(\boldsymbol{x})}{\sigma_t}\right),\ t = 0, \ldots, T, \quad \infty = \sigma_0 > \sigma_1 > \ldots > \sigma_T \approx 0$$

- The CE optimization problem is then solved iteratively to get a good ISD
- $p(x;\hat{v}^{(t-1)})$ is chosen as the reference distribution for the CE procedure at level t.

Fitting the Bernoulli mixture model

- For updating the parameters of the Bernoulli mixture model, we use a modified version of the expectation-maximization (EM) algorithm that works with weighted samples (Geyer et.al., 2019).
- A conjugate prior is used to smooth the weighted samples and help mitigate the overfitting issue.
- A model selection technique is employed to estimate the number of clusters K in the mixture.

Numerical examples

Toy example (two terminal connectivity problem)

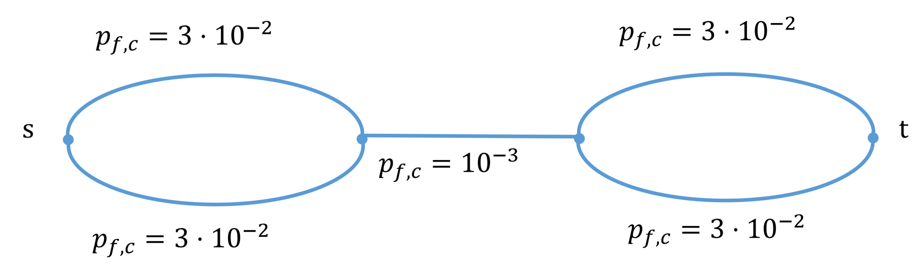


Fig. 1. Topology of the network.

- $g(X) = \mathbf{conn}(s,t)$
- The failure probability p_f is equal to 0.0028.
- The optimal ISD has three major modes.

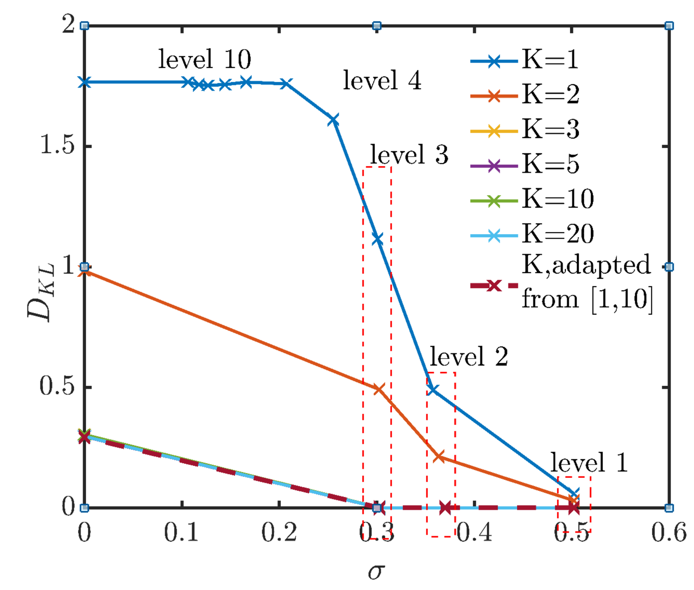


Fig. 2. Performance of the improved CE estimator of the toy example (sample size: 10^5)

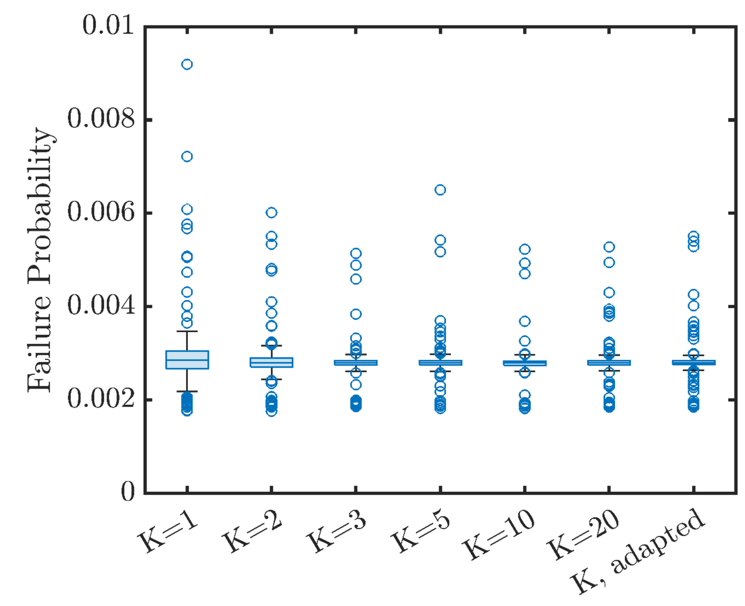


Fig. 3. Boxplot of the improved CE estimator of the toy example (sample size: 10^3)

IEEE39 DC flow model

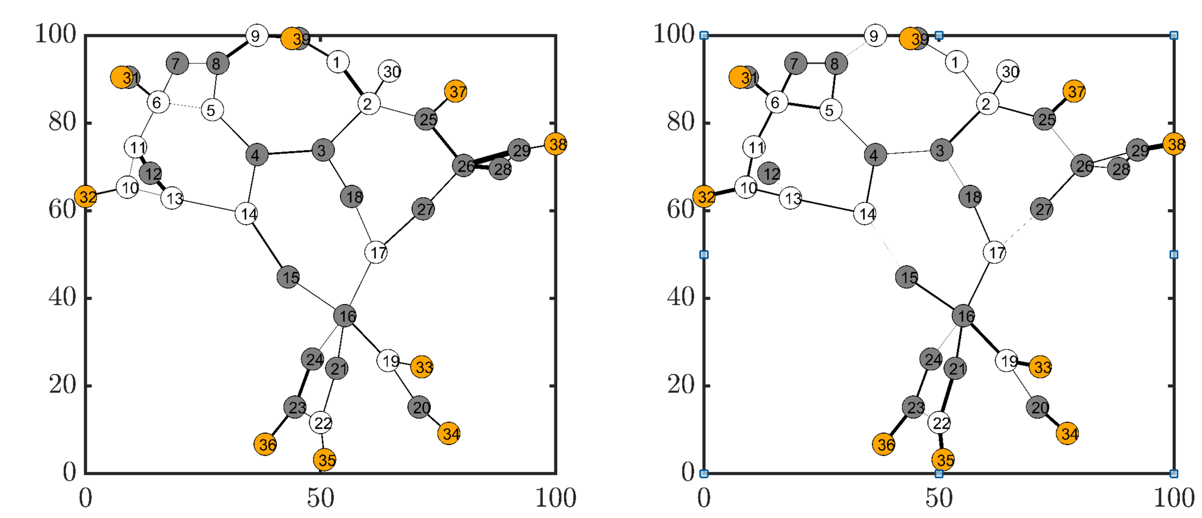


Fig. 4. Topology of the IEEE39 network, with edge thickness proportional to their estimated capacities (left) and reactance (right).

$g(X) = 50\% - Loss_{norm}(X)$

$X \sim i.i.d$ Ber

State	fail	survive
Prob.	0.01	0.99

- The failure probability p_f is equal to $2.16 \cdot 10^{-4}$.
- The optimal ISD has one major mode.

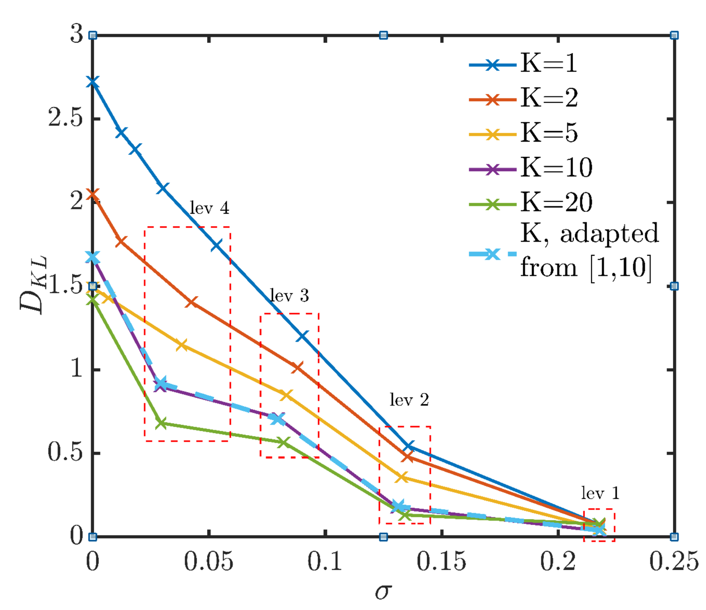


Fig. 5. Performance of the improved CE estimator of the IEEE39 DC flow model (sample size: 10^5)

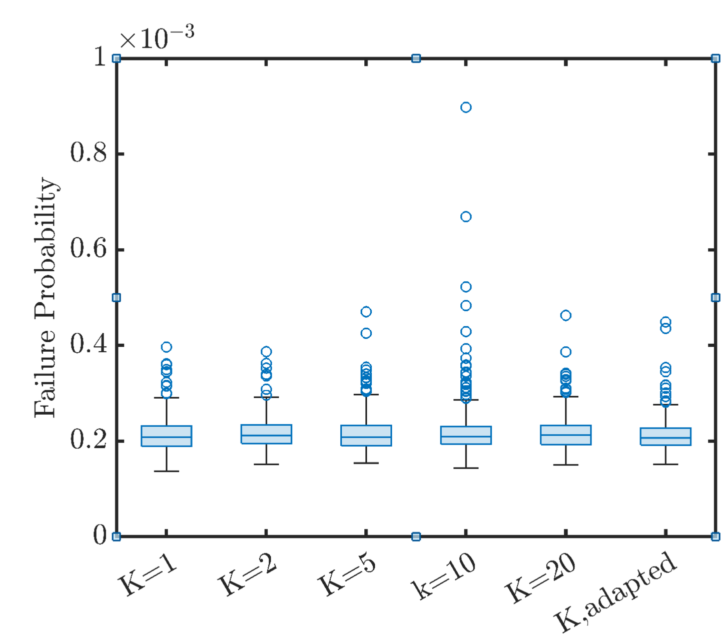


Fig. 6. Boxplot of the improved CE estimator of the IEEE39 DC flow model (sample size: 10^3)

- The improved CE method with single Bernoulli model (K=1) converges the slowest.

Conclusions

- For problems with moderate number of components, the improved cross entropy method using Bernoulli mixture outperforms the one using the independent Bernoulli model.
- The improvement can be marginal when the system components are not strongly dependent conditional on the system failure.

Reference

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- Papaioannou I, Geyer S, Straub D. Improved cross entropy-based importance sampling with a flexible mixture model[J]. Reliability Engineering & System Safety, 2019, 191: 106564.
- Rubinstein R. The cross-entropy method for combinatorial and continuous optimization[J]. Methodology and computing in applied probability, 1999, 1(2): 127-190.