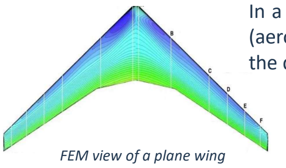
A parametric benchmark problem for multidisciplinary design optimization under uncertainty

Amine Aziz-Alaoui [1,2], Olivier Roustant [2], Matthias De Lozzo [1]
[1] IRT Saint Exupéry [2] INSA Toulouse

1. Introduction

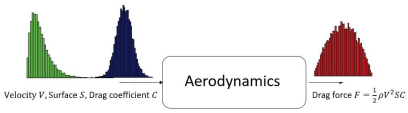
- The introduction of **uncertain parameters** in multidisciplinary design optimization (MDO) problems is an active research topic [1] .

  But it raises two main difficulties:
  - considerably increases the **complexity** of the problem due to repeated runs of the deterministic case
  - can only be tested on **industrial use-cases**



FEM view of a plane wing

In a plane wing MDO problem with 2 disciplines (aero, structure), an uncertain variable can be the drag coefficient:



- **Objective:** To assess the efficiency of uncertain quantification methods in MDO, build a parametric benchmark problem:
  - handling different problem sizes (**scalability**),
  - handling **different distributions** for the uncertain variables,
  - **compatible** with the whole **UMDO process**,
  - providing an analytical solution.

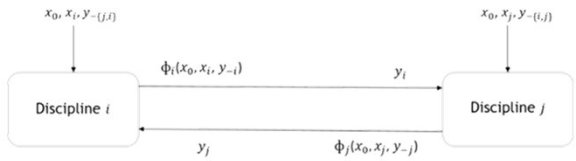
2. Multidisciplinary design optimization

- **Optimization** problem:

$$\text{minimize } f(x) \text{ subject to } g(x) \leq 0$$

- **Multidisciplinary design optimization** problem:

f and g result from the evaluation of N coupled *disciplines* $\phi_1, \ldots, \phi_N$:



where $x = (x_0, x_1, \ldots, x_N)$ are the design variables and $y = (y_1, \ldots, y_N)$ the coupling ones.
➔ One of the challenges of MDO is to solve the coupling system $y = \phi(x, y)$.

- **MultiDisciplinary Feasible (MDF)** formulation of the MDO problem:

If ϕ is C^1, there exists a C^1 function c such that $y = c(x)$**.** Then,

$$\text{minimize } \tilde{f}\,(x, c(x))$$
$$\text{subject to } g_0(x, c(x)) \leq 0 \text{ and } g_i(x_0, x_i, c_i(x)) \leq 0,\ i \in \{1, \ldots, N\}$$

In practice, $c(x)$ is approximated with a point-fixed algorithm.

3. A parametric benchmark for robust MDO

- MDF formulation of a **robust MDO problem**:

$$\text{minimize } \mathbb{F}\big[f\big(x_0, U_0, c(x, U)\big)\big]$$
$$\text{subject to}$$
$$\mathbb{G}_0\big[\ g_0\big(x_0, U_0, c(x, U)\big)\big] \leq 0 \text{ and } \mathbb{G}_i\big[g_i\big(x_0, x_i, c_i(x, U)\big)\big] \leq 0,\ i \in \{1, \ldots, N\}$$

where $U_0, U_1, \ldots, U_N$ are independent centered random variables with covariance matrices $\Sigma_0 \ldots \Sigma_N$ and $\mathbb{F}$and $\mathbb{G}$ are deterministic statistics.

- [2] proposed a parametric MDO problem to which we added random variables at the disciplinary levels:

$$\text{minimize } \mathbb{F}\big[x_0^T x_0 + \textstyle\sum_{i=1}^N Y_i^T Y_i\big]$$
$$\text{subject to } \mathbb{G}_0\big[\ g_0\big(x_0, U_0, c(x, U)\big)\big] \leq 0$$
$$\text{where } Y_i = c_{yi} - C_{y_i x_0} x_0 - C_{y_i x_i} x_i + \textstyle\sum_{i=1}^N C_{y_i y_j} Y_i + U_i$$

- **Our contributions**:
  - Scale the coupling variables to the unit hypercube.
  - Rewrite the MDO problem as a quadratic programming (QP) problem:

$$\text{minimize } \frac{1}{2} x^T Q x + c^T x$$
$$\text{subject to } Ax \leq b$$

  - Find the conditions on the variable dimensions for a unique global minimum x^*.
  - Reduce the robust MDO problem to a QP problem when $\mathbb{F}$ is the expectation and $\mathbb{G}$ is a margin or a probability(1):

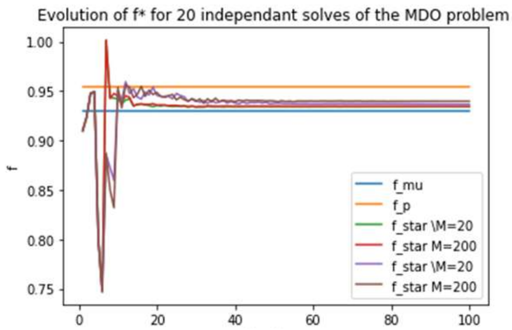
$$\text{minimize } \frac{1}{2} x^T Q x + c^T x$$
$$\text{subject to } Ax \leq b + e$$

  - Estimate the statistics with **Monte Carlo (MC)** or **Taylor series expansion (TSE)**.

(1) margin: $\mathbb{G}[g(x,U)] = \mathbb{E}[g(x,U)] + \kappa\sqrt{\mathbb{V}[g(x,U)]}$; probability: $\mathbb{G}[g(x,U)] = \mathbb{P}[g(x,U) \geq 0] - \epsilon$.

4. Numerical experiments

- **Problem**:
  - $N = 2, n_{x_0} = 1, n_{x_1} = 2, n_{x_2} = 2, n_{y_1} = 3, n_{y_2} = 3.$
  - $\Sigma = \sigma \times I_{n_y}$ with $\sigma = 0.01, \kappa = 2,\ \epsilon = 0.01.$
- **Numerical settings:**
  - Sampling: 20 or 200 Monte Carlo realizations, repeated 20 times.
  - First-order linearization: analytical derivatives.
  - Second-order linearization: analytical derivatives + finite differences.
  - Optimization algorithm: COBYLA.



MC	$\overline{\frac{\lVert x^* - x^{\mathbb{P}} \rVert}{\lVert x^{\mathbb{P}} \rVert}}$	$\overline{\frac{\lVert f^* - f^{\mathbb{P}} \rVert}{\lVert f^{\mathbb{P}} \rVert}}$	$\overline{\lVert \mathbb{G}^* \rVert}$	$\sigma \left\lVert \frac{x^* - x^{\mathbb{P}}}{x^{\mathbb{P}}} \right\rVert$	$\sigma \left\lVert \frac{f^* - f^{\mathbb{P}}}{f^{\mathbb{P}}} \right\rVert$
			$\mathbb{G} = \mathbb{P}(g(x,U) \geq 0) - \epsilon$		
$M = 20$	0.0786	-0.0139	0.0323	0.0437	0.0117
$M = 200$	0.0517	-0.0126	0.0208	0.0234	0.0052
			$\mathbb{G} = \mu_g + \kappa\sqrt{\text{diag}(\Sigma_g)}$		
$M = 20$	0.0218	0.0045	29.2130	0.0035	0.0130
$M = 200$	0.0162	0.0050	29.139	0.0060	0.0014
TSE	$\frac{\lVert x^* - x^{\mu} \rVert}{\lVert x^{\mu} \rVert}$	$\frac{\lVert f^* - x^{\mu} \rVert}{\lVert f^{\mu} \rVert}$	$\lVert \mathbb{G}^* \rVert$	-	-
			$\mathbb{G} = \mu_g + \kappa\sqrt{\text{diag}(\Sigma_g)}$		
1^{st} order	0.0141	0.0046	29.113	-	-
2^{nd} order	0.0142	0.0048	29.112	-	-

- Estimation of the reference result with less than 4% rate of error.
- Complexity when the constraint is a probability.

5. Perspectives

- Deepen the results of MC and TSE: probability distributions, variance rate, sampling size, …
- Test other estimation techniques: MLMC, FORM/SORM, importance sampling, stochastic algorithms, …
- Extension to decoupled and bi-level formulations.

[1] W. Yao, X. Chen, W. Luo, M. van Tooren, and J. Guo, *Review of uncertainty-based multidisciplinary design optimization methods for aerospace vehicles*, Progress in Aerospace Sciences, 47 (6), 2011.
[2] N.P. Tedford and J.R.R.A. Martins, *Benchmarking multidisciplinary design optimization algorithms*, Optimization and Engineering, 11, 2010.
[3] M.K. Kozlov, S.P. Tarasov, and L.G. Khachiyan, *The polynomial solvability of convex quadratic programming*. USSR Computational Mathematics and Mathematical Physics, 20 (5), 1980.
[4] F. Delbos, J.C. Gilbert, *Global linear convergence of an augmented Lagrangian algorithm to solve convex quadratic optimization problems*, Journal of Convex Analysis, 12, 2005.

IRT SAINT EXUPÉRY
GEMSEO An open source Python library for MDO
Université de Toulouse
Σ_d mitt
INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE

Amine Aziz-Alaoui
PhD Student
amine.aziz-alaoui@irt-saintexupery.com