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Gaussian process based reachability analysis

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Abstract:

In the context of dynamic systems, reachability analysis computes the set of all possible trajectories of the states that a dynamical system can reach, under model uncertainties and perturbations, starting from a set of initial conditions [2]. Reachability analysis helps to determine: 1) whether the states of the dynamical system ever can belong to a set characterised as unsafe (by definition) in finite or infinite time; 2) whether the trajectories of the dynamical system can reach a goal region around a set point; and 3) the parameter sets for the dynamical system which satisfy given requirements, among others.

Set-propagation techniques that depend on ellipsoids, Taylor series method, zonotopes, level sets and intervals and simulation-based techniques that depends on numerical solution to ODEs are popular approaches to determine reachable sets. Methods making use of support functions have also been investigated. A data-driven methodology to determine the reachability set which is based on matrix zinotopes has been proposed recently in [1]. This approach computes over-approximative reachability sets directly from the noisy data with a given system model.

In this work, we introduce a Gaussian process(GP) based approach to compute the reachability set. Gaussian process [3] is a nonparametric stochastic process over functions, it inferences highly nonlinear latent function from (noisy) observations in a Bayesian way. Gaussian process is a generalisation of the multivariate Gaussian distribution to an infinite number of dimensions. Gaussian process has been frequently used for solving problems including nonlinear regression and classification, not only due to the existence of Gaussian distributions in the real world, but also properties in rendering the mathematical manipulation tractable. In the proposed approach, the finite time simulation of the dynamical model will be regarded as a 'black-box' which maps certain set of inputs (including initial conditions, model uncertainties and control input) to the output which is the set of states at a specific time.

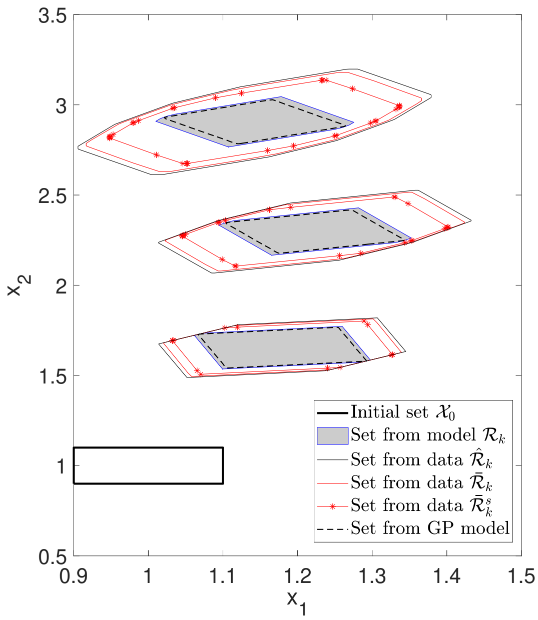
Given a dynamical system $\dot{x} = f(x(t), u(t), \delta(t))$, where t is the time, x is the state, u is the input, and δ is the model uncertainties, the reachable set at a certain instant of time T is

$$\mathcal{R}(T) = \left\{ x(T) = \int_0^T f\left(x(\tau), u(\tau), \delta(\tau)\right) d\tau \;\middle|\; x(0) \in \mathcal{X}_0,\ u \in \mathcal{U},\ \delta \in \Delta \right\},$$

where $\mathcal{X}_0$, $\mathcal{U}$, Δ are the bounded sets for possible initial states, the control input and model uncertainties.

The GP model is constructed from a training dataset $\mathcal{L}$ which consists of N samples of $\theta = \{x(0), u, \delta\}$ and the their observations $y(\theta)$, for a given T. The trianing dataset $\mathcal{L} = \{\mathcal{D}, \mathbf{y}\}$, where $\mathcal{D} = \{\theta_1, \theta_2, \cdots, \theta_N\}$, $\mathbf{y} = \{y(\theta_1), y(\theta_2), \cdots, y(\theta_N)\}$. A Gaussian posterior predictive distribution at any test θ_* is specified by mean $\mu(\theta_*)$ and covariance $\Sigma(\theta_*)$, more information can be find in [3].

The performance of the proposed algorithm is tested with an abstract, stable linear time-invariant five-dimension system used in [2]. Further, we obtain reachability set results from three existing available methods, LTI-Reachability Algorithm, LTI-Constrained-ReachabilityAlgorithm and LTI-Side-Info-Reachability Algorithm, as proposed in [1], and are given here for comparison. The initial set is chosen to be $\mathcal{X}_0 = \left\{[x_i], i = 1, \cdots, 5 \;\middle|\; x_i \in [0.9, 1.1]\right\}$, control input set is $\mathcal{U} = \left\{[u_i], i = 1, \cdots, 5 \;\middle|\; u_i \in [9.98, 10.02]\right\}$, and there is no model uncertainties. We compute the readable sets when $T = 0.05$, $T = 0.10$ and $T = 0.15$.



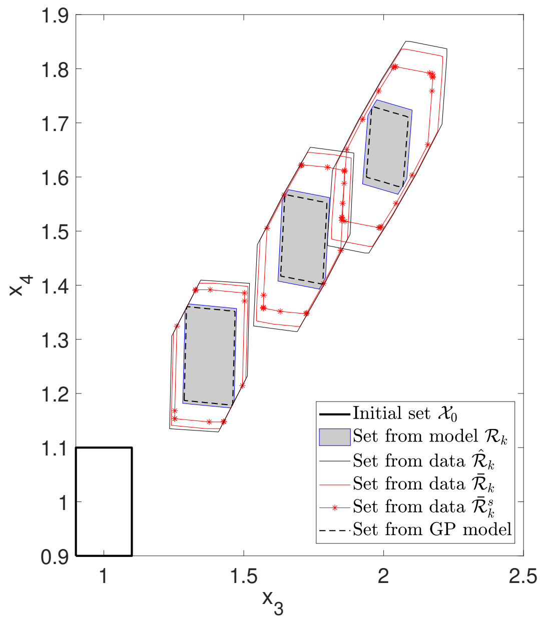


Figure 1: Projection of the reachable sets on x_1-x_2 plane

Figure 2: Projection of the reachable sets on x_3-x_4 plane

Figures 1 and 2 give the projections of the reachable sets of the LTI system on x_1-x_2 and x_3-x_4 state plane respectively. The reachable sets are computed via LTI-Reachability Algorithm($\hat{\mathcal{R}}_k$), LTI-Constrained-ReachabilityAlgorithm($\bar{\mathcal{R}}_k$), LTI-Side-Info-Reachability Algorithm($\bar{\mathcal{R}}_k^s$) and GP based algorithm (black dashed line) from noisy input-state data. The black rectangles in the left bottom of Figures 1 and 2 indicate the initial set, and then from the bottom to the top, three groups of reachable sets are for $T = 0.05$, $T = 0.10$ and $T = 0.15$, respectively. From both projections we can see that the algorithms provided in [1] over-approximate the reachable sets, and the Gaussian process based algorithm proposed in this work slightly under-approximates the actual reachable sets given in blue($\mathcal{R}_k$). The execute time of those three algorithms from [1] is 1.8142 minutes in total, and the execute time of GP based algorithm is 0.5724 minutes (less than 1.8142/3 minutes).

In this work, the five dimension system states are modelled by five independent GPs. Future focus will be using the multiple output Gaussian process rather than the single GP, in order to improve the model accuracy and reduce the computational complexity and employ the approach to complex aerospace simulation models.

References

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Short biography – Ke Wang received his bachelor degree from NWPU(China) and master degree from BUAA(China) both in aerospace engineering. After that, he joined Unversity of Exeter and is working on 'Bayesian approach for robustness analysis of complex models'. His PhD is funded by a European Space Agency project, for which SENER Aerospacial is prime contractor. His research interests include spacecraft formation flight, space trajectory design and optimization, and machine learning.